PHYSICS TEXTBOOK

Yakir Aharonov
Daniel Rohrlich

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Quantum Paradoxes

Quantum Theory for the Perplexed

Yakir Aharonov and Daniel Rohrlich

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Preface

Quantum Paradoxes is a series of studies in quantum theory. Each chapter begins with a paradox motivating the study, in the rest of the chapter, of a fundamental aspect of the theory. We hope that this style in physics – progress through paradox – will rub off on readers. The studies, taken together, set out a new interpretation of quantum theory.

Elements of this interpretation include topological phases (the Aharonov-Bohm effect and its generalizations), "modular" variables, nonlocal measurements and relativistic causality, time-symmetric boundary conditions, measurement of the quantum wave, "weak" measurements and "weak" values, and new axioms for quantum theory. A treatment of "quantum measurements", starting in Chap. 7, plays an important role in the book. Indeed, measurement is so important that many of the works cited in the book can be found in the anthology *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton: Princeton U. Press), 1983; these citations include a note "reprinted in WZ" with page numbers.

For whom is this book written? It is designed for physics students, physicists and philosophers of science with an interest in fundamental aspects of quantum theory. The first two chapters of *Quantum Paradoxes* do not require prior knowledge of quantum theory, and Chaps. 3–4 introduce basic notions of states, observables and quantum phases, so students can use the book even during a first course in quantum mechanics. It is not, however, a substitute for such a course.

Each chapter ends with a problem set. Problems marked with an asterisk (*) are, in general, less straightforward than others.

It is a pleasure to thank those who have helped us write this book. We are indebted to colleagues (including students) who read parts of the book at one stage or another, most especially Philip Pearle and Fritz Rohrlich, and to Elisabeth Warschawski for much encouragement and technical support. We thank Shula Volk for opening her ceramics studio to us at odd hours of the day and night, as reported in Sect. 2.1. We also acknowledge support from the Giladi Program of the Israeli Ministry of Absorption and from the Ticho Fund.

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1 The Uses of Paradox

On November 9, 1919, *The New York Times* reported solar eclipse observations confirming a prediction of Einstein's general theory of relativity: rays of starlight bend near the sun. It also reported that when Einstein sent his theory to the publishers, "he warned them that there were not more than twelve persons in the world who would understand it ...." Was there a time when only "twelve wise men" understood the general theory of relativity? "I do not believe there ever was such a time," commented Feynman. "There might have been a time when only one man did, because he was the only guy who caught on, before he wrote his paper. But after people read the paper a lot of people understood the theory of relativity in some way or other, certainly more than twelve. On the other hand, I think I can safely say that nobody understands quantum mechanics." [1]

What is the problem with quantum mechanics? It is a spectacularly successful theory. It governs the structure of all matter. Measurements of Planck's constant are accurate to better than a part in a million, and still more accurate measurements confirm predictions of quantum electrodynamics. But along with the spectacular successes of quantum mechanics come spectacular difficulties of interpretation. "Do not keep saying to yourself, if you can possibly avoid it, 'But how can it be like that?"' Feynman continued, "because you will get 'down the drain', into a blind alley from which nobody has yet escaped. Nobody knows how it can be like that."

We can stop asking ourselves, "But how can it be like that?" We may indeed despair of asking a question that Einstein, Schrödinger and Feynman could not answer. But we cannot stop *using* quantum mechanics. So the problem is that everybody uses quantum mechanics and nobody knows how it can be like that. Our relationship with quantum mechanics recalls a Woody Allen joke:

> This guy goes to a psychiatrist and says, "Doc, my brother's crazy – he thinks he's a chicken! And, uh, the doctor says, "Well, why don't you turn him in?" And the guy says, "I would, but I need the eggs!"

We say, "Quantum mechanics is crazy – but we need the eggs!"

Such a relationship with quantum mechanics is paradoxical. In this book, we will not be satisfied to have a paradoxical relationship with quantum mechanics. We will not stop asking, "How can it be like that?" But we will *use* paradox repeatedly in order to understand quantum mechanics better.

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1.1 Paradox in Physics

We will use paradox to probe quantum mechanics. Can paradox be useful? The history of physics shows *how* useful. As Wheeler [2] put it, "No progress without a paradox!" In this section, we define and classify physics paradoxes; the next sections present examples of each class.

A paradox is an argument that starts with apparently acceptable assumptions and leads by apparently valid deductions to an apparent contradiction. Since logic admits no contradictions, either the apparently acceptable assumptions are not acceptable, or the apparently valid deductions are not valid, or the apparent contradiction is not a contradiction. A paradox is useful because it can show that something is wrong even when everything appears to be right. It does not show *what* is wrong. But something is wrong – something we thought we understood – and a paradox moves us to reexamine the argument until we find out what is wrong.

We can classify physics paradoxes according to what is wrong. There are three broad classes: "errors", "gaps" and "contradictions".

Many paradoxes arise from errors. An error in logic or in our understanding of the laws of physics easily leads us to an apparent contradiction. Our error may be simple or it may be subtle, but it is just an error; once we recognize it, we have resolved the paradox. What distinguishes the first class is that these paradoxes do not arise from any flaw in the theory. In the special theory of relativity, for example, erroneous assumptions about simultaneity lead us to paradox. (See Sect. 1.2.) Resolving the paradox, we improve our understanding of special relativity, but we do not improve the theory. Another example of a paradox arising from an error is Einstein's clock-in-the-box paradox. (See Sect. 2.4.) Einstein made an error and arrived at an apparent contradiction in quantum theory. The resolution of the paradox came as a surprise, but it did not show quantum theory to be flawed in any way.

Other paradoxes *do* show a physical theory to be flawed. A gap in physical theory is a flaw. As an example of a gap, consider Wheeler's paradox of black hole entropy. According to the general theory of relativity, nothing can escape a black hole. We, as outside observers, can measure the electric and gravitational fields of a black hole, and hence its charge and mass (and angular momentum); but we have no other access to a black hole. So a black hole at rest has only three properties: charge, mass and angular momentum. Such a simple physical system can hardly have much entropy. Now suppose a complicated physical system, containing a lot of entropy, falls into a black hole. What happens to the entropy? Apparently it vanishes. But vanishing entropy violates the second law of thermodynamics. Wheeler told his student Bekenstein about this paradox:

> The idea that a black hole had no entropy troubled me, but I didn't see any escape from this conclusion. In a joking mood one day in my office, I remarked to Jacob Bekenstein that I always feel like a criminal when I put a cup of hot tea next to a glass of iced tea and then let the two come to a common temperature, conserving the world's energy but increasing the world's entropy. My crime, I said to Jacob, echoes down to the end of time, for there is no way to erase or undo it. But let a black hole swim by and let me drop the hot tea and the cold tea into it. Then is not all evidence of my crime erased forever? This remark was all that Jacob needed [3].

Bekenstein [4] proposed that a black hole has entropy proportional to the square of its mass. If any physical system falls into a black hole, the mass of the black hole increases – and hence the entropy. He demonstrated that the increase in entropy is at least as great as the entropy of the infalling system, thus corroborating the second law and resolving Wheeler's paradox.

Wheeler's paradox indicated a flaw – but not a fatal flaw – in general relativity and thermodynamics. The resolution of the paradox did not invalidate either theory. The apparent contradiction between the theories arose from a gap in thermodynamics – we didn't know how to extend the concept of entropy to black holes – and Bekenstein's proposal filled the conceptual gap. Another paradox in the second class came, in turn, from Bekenstein's proposal: if thermodynamics extends to black holes, then black holes must emit as well as absorb heat. But nothing can escape a black hole! This paradox, too, arose from a conceptual gap, as Hawking discovered: one consequence of the uncertainty principle is that black holes *radiate* [5]. Many such paradoxes appear in this book.

A contradiction in physical theory is a fatal flaw. Paradoxes in the third class are associated with revolutions in physics, because they indicate that the physical theory behind the paradox is wrong. Bohr faced such a paradox in 1911. In that year, Rutherford reported experiments on neutral atoms, showing that the positive charges in atoms – but not the negative charges (electrons) – are concentrated in nuclei. According to classical theory, such atoms should be unstable: like all accelerating charges, the electrons should radiate energy, and fall into the nuclei. Matter should collapse in a split second. So why is matter stable? Bohr realized that this paradox had no resolution in classical physics. Only a new physical theory – quantum theory – could resolve it. The only resolution was a revolution.

The paradox arose for Bohr as a contradiction between physical theory and experiment. Especially useful are paradoxes that arise as contradictions *within* physical theory. Such a paradox can show that a physical theory is wrong even when no experiment contradicts it. The paradox then starts us searching for a new theory. (See Sects. 1.4 and 2.2.)

1.2 Errors

Every student of special relativity encounters the Twin Paradox [6]. Here is a Triplet Paradox. Dumpy, Grumpy and Jump – identical triplets wearing synchronized wristwatches – once lived together happily at home. But Grumpy got mad at Dumpy and decided to move to another city. When he arrived, his watch was still synchronized with his brothers' watches, because he travelled very slowly compared to the speed of light. (In this paradox we set the speed of light to 1000 m/s.)

A month later, Jump decided to visit Grumpy. Dumpy accompanied Jump to the train station, and Jump took a seat in the train. Then the train accelerated, within a second, to 100 m/s. At the end of this second, Jump's cabin passed Dumpy on the platform. Jump and Dumpy glanced at each other through a cabin window and noticed that their watches still showed the same time (to within a second). Hence Jump did not age appreciably during the acceleration. For the rest of the trip, the train's speed and direction were constant. When it arrived, it stopped within a second.

Dumpy and Grumpy expected that Jump would be slightly younger than them when he arrived, and that his watch would lag behind their watches, for Jump had been moving fast

relative to them. But Jump expected the opposite: Dumpy and Grumpy would be slightly younger, and their watches would lag behind his. He told himself, "After one second of acceleration, Dumpy's watch and mine showed the same time (to within a second); and Dumpy's and Grumpy's watches were still synchronized. Afterwards, the inertial reference frame of Dumpy and Grumpy moved fast relative to mine; so time passed more slowly for them, and their watches now lag behind mine." When he arrived, he discovered that his watch lagged Grumpy's by about half a minute! On the one hand, Jump's expectation should be just as correct as that of his brothers; there can be no preferred frame in special relativity. On the other hand, Jump and his brothers cannot all be correct. So special relativity contradicts itself!

The Triplet Paradox belongs to the class of errors in that it does not arise from any flaw or misconception in the special theory of relativity. It arises, rather, from incorrect intuition. We can often use paradoxes in this class to improve our intuition.

1.3 Gaps

In 1856, Clausius stated the second law of thermodynamics as follows: heat cannot flow from a colder body to a hotter body without an accompanying process (i.e. work). Clausius regarded the second law as exact, and tried to derive it from the laws of mechanics. In 1871 he published a paper [7] in which he offered a mechanical explanation of the second law. He did not know that Boltzmann had published much the same explanation five years earlier [8]. Boltzmann (who, like Clausius, regarded the second law as exact) was quick to claim priority [9]. Yet Clausius did not wholly concede [10]. Maxwell was amused. "But it is rare sport to see those learned Germans contending for the priority of the discovery that the 2nd law of $\theta\Delta cs$ is the Hamiltonische Princip ... ." he wrote. "The Hamiltonische Princip, the while, soars along in a region unvexed by statistical considerations ... ." [11]. Boltzmann and Clausius were both wrong. The second law has no mechanical explanation; it is statistical.

What made Maxwell so sure that the second law is statistical? In 1859 he had calculated that the distribution of molecular speeds in any gas, hot or cold, would range from zero to infinity. (Molecules were still an untested hypothesis at the time.) In 1867 he had considered the following thought experiment. Gas fills a sealed, insulated box, divided by a diaphragm. The gas is hot on one side of the diaphragm and cold on the other side; yet there are fast molecules in the cold gas and slow molecules in the hot gas. "Now conceive a finite being who knows the paths and velocities of all the molecules by simple inspection but who can do no work except open and close a hole in the diaphragm by means of a slide without mass." The being opens and closes the hole in such a way that fast molecules in the cold gas enter the hot gas and slow molecules in the hot gas enter the cold gas. Energy gradually flows from the cold gas to the hot gas. After many molecules have crossed through the hole, "the hot system has got hotter and the cold colder and yet no work has been done, only the intelligence of a very observant and neat-fingered being has been employed" [12]. The "neat-fingered being" soon had a name: "Maxwell's demon".

Maxwell's demon violates the second law of thermodynamics, as formulated by Clausius: it does no work, yet it causes heat to flow from a cold gas to a hot gas. It does not, however, violate the laws of mechanics. Hence the second law cannot be a mechanical law. Maxwell's

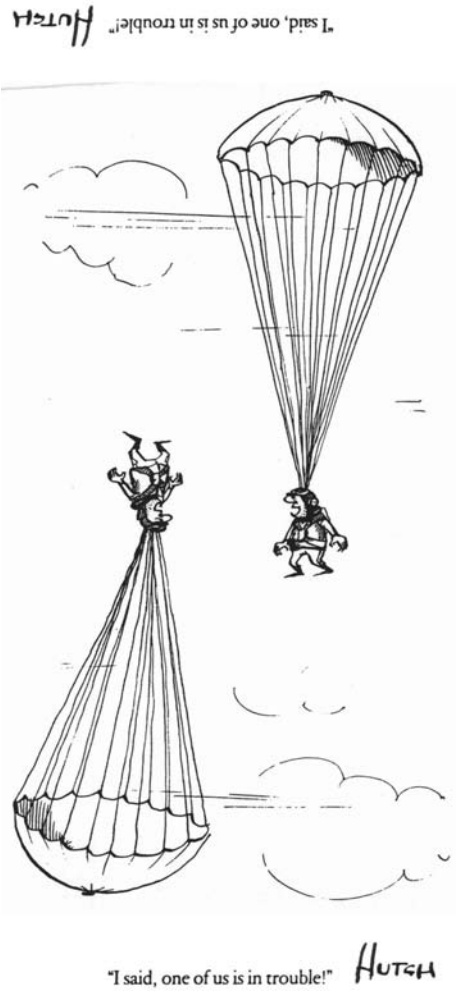

Figure 1.1: Two opposite "arrows of space". [With thanks to Stuart M. Hutchison and Princeton Tiger Magazine.]

thought experiment was a paradox for Clausius's formulation; it does not disprove the second law, but it shows that the second law can only be a statistical law.

Another formulation of the second law states that the entropy of a closed system always tends to increase to thermal equilibrium. But this formulation, too, leads to a paradox. It assumes an arrow of time, relative to which entropy tends to increase. But what if there is no arrow of time? What if the "arrow of time" is no more intrinsic than the "arrow of space" defined by gravity? (See Fig. 1.1.) Suppose that two sealed, insulated boxes are filled with gas, e.g. helium in one box and neon in the other, and at time $t = 0$, neither gas is at thermal equilibrium. Now on the one hand, if the boxes are perfectly insulated, they could contain two opposite arrows of time. Assume that the gases have contrary evolutions: the entropy of the neon increases in time while the entropy of the helium *decreases* in (the same) time. Such an assumption is plausible since the laws of mechanics are invariant under time reversal and the

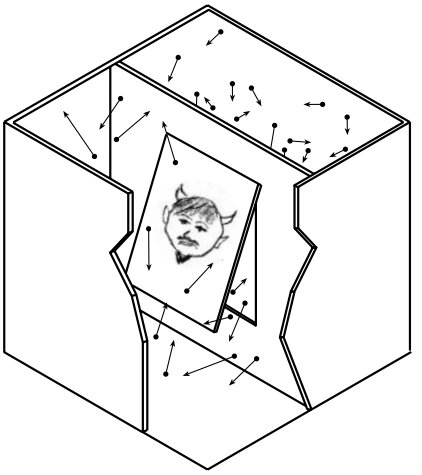

Figure 1.2: Maxwell's demon as a trapdoor.

boxes do not interact. On the other hand, suppose the boxes *do* interact, with an interaction that is independent of time; assume that the position and momentum of each atom at $t = 0$ is the same as before. According to the second law, the combined entropy of the two gases always tends to increase; that is, any perturbation of the helium atoms, however small, will destroy the precise coordination of their positions and momenta that allows their entropy to decrease. So in the evolution of the two gases after $t = 0$, their total entropy increases. But the same reasoning applies in reverse to the evolution of the gases before $t = 0$: their total entropy must *decrease* until $t = 0$. Extrapolation forwards from $t = 0$ implies that the neon (with its increasing entropy) overwhelms the helium; extrapolation backwards from $t = 0$ implies that the helium overwhelms the neon. This paradox shows that the second law contains no arrow of time. (See also Chap. 10.)

The second law is *almost* exact, i.e. the probability of a significant violation is very small. Maxwell's demon can violate the second law, yet the probability of a significant violation is very small. Still, after Maxwell, the demon turned up in new paradoxes. The demon kept turning up, because it is easier to imagine a demon that can violate the second law significantly, than to prove that it can't. For example, in Fig. 1.2 the demon is a trapdoor that apparently allows only fast molecules of the cold gas to enter the hot gas. In 1914, Smoluchowski showed that this demon fails to violate the second law significantly because the trapdoor itself thermalizes, eventually opening and closing in random fluctuations [13]. More recent paradoxes allow Maxwell's demon to measure and compute. Their resolution involves an application of information theory to thermodynamics [14].

All the paradoxes in this section belong to the class of gaps; they show up flaws or gaps in how we understand the second law, but do not invalidate it. The resolutions of these paradoxes correct our formulation of the second law and extend the concepts we use to apply it, but do not contradict the formalism of thermodynamics.

1.4 Contradictions

Maxwell's equations imply that a changing electromagnetic field in empty space propagates as a wave with constant speed c. On the face of it, this implication contradicts Newton's mechanics. According to Newton, if we run after a light wave, its speed (relative to us) decreases. Velocities add as vectors: if the velocity of a light ray with respect to Alice is $\mathbf{v}_A$ and the (nonzero) velocity of Alice with respect to Bob is $\mathbf{v}_{AB}$, then the velocity of the ray with respect to Bob is $\mathbf{v}_A + \mathbf{v}_{AB}$. The ray cannot have the same speed for Alice and for Bob. What, then, corresponds to c? Physicists of Maxwell's time assumed that electromagnetic waves propagate through a medium, the "aether", and what corresponds to c is the speed of the wave relative to the aether.

At first, the aether was a plausible assumption. Even before Maxwell, physicists assumed that light propagates through an aether. Every wave known to them, from ripples in water to sound in air, propagated through some medium. Fresnel showed in 1818 that an aether at absolute rest, unaffected by the earth's motion through it, would be consistent with the "aberration effect", a seasonal shift in the apparent positions of stars in the sky. Over the rest of nineteenth century, however, the aether became less and less plausible. In 1887, Michelson and Morley measured the speed of light parallel and perpendicular to the earth's motion, and found no difference.[1] Hence the aether and the earth must move together. Or else the earth is at absolute rest – Copernicus was wrong after all!

Aether was a paradox. But since an aether at absolute rest made sense of c (and defined the "absolute space" that Newton had postulated), most physicists chose tacitly to live with it. They then had to explain the contradictory experiments.

For Einstein, the paradox was different. He concluded early on (even without the Michelson–Morley experiment) that there *is* no aether. He was then left with the contradiction between Newton's mechanics and electromagnetism. At age 16, Einstein formulated the paradox as follows:

> If I pursue a beam of light with the velocity c (velocity of light in a vacuum), I should observe such a beam of light as an electromagnetic field at rest though spatially oscillating. There seems to be no such thing, however, neither on the basis of experience nor according to Maxwell's equations. From the very beginning it appeared to me intuitively clear that, judged from the standpoint of such an observer, everything would have to happen according to the same laws as for an observer who, relative to the earth, was at rest. For how should the first observer know, or be able to determine, that he is in a state of fast uniform motion? [15]

Newton proved that his laws of mechanics are the same for all observers in uniform (rectilinear) motion; and Maxwell [16] realized that – apart from the aether – electromagnetism is the same for all observers in uniform motion. If there is no aether, then the laws of mechanics and electromagnetism must be the same for all observers in uniform motion. The paradox was that in Newton's mechanics, which relates such observers by Galilean transformations, the speed of light is not a constant; in Maxwell's electromagnetism, which relates such observers by Lorentz

[1] To show that the basement laboratory of Michelson and Morley did not trap aether, Morley and Miller later repeated the measurements on a hilltop.

transformations, the speed of light *is* a constant, as it is in experiment. Years later, Einstein resolved the paradox by modifying Newton's mechanics so that Lorentz transformations, rather than Galilean transformations, relate observers in uniform motion.

Indeed, paradox can be useful but, as this example shows, the paradox has to be the *right* paradox. Where other physicists saw a contradiction between physical theory and experiment, Einstein (and to an extent Poincaré) saw a contradiction within physical theory. What is striking in this example is how ready Einstein was to discard the aether assumption when it had become implausible (but still accepted by all other physicists) and to face a fundamental contradiction in physical theory. He was thus able to identify the right paradox behind the wrong paradox, and later to resolve it.

1.5 Overview of the Book

In this book we have two goals. Our primary goal is a deeper understanding of all fundamental aspects of quantum mechanics. The first chapters do not assume prior knowledge of quantum mechanics. Chapter 2 discusses uncertainty and consistency in quantum theory, without the formalism, and Chap. 3 introduces the notion of a quantum state, while discussing the completeness of quantum theory. Chapter 4 introduces the quantum phase and probes its unique role in the theory with the help of Schrödinger's equation. Chapters 5 and 6 apply the phase in unconventional ways and raise unconventional questions about quantum mechanics. Chapters 7–11 present a theory of quantum measurements, a powerful tool for probing quantum mechanics. Chapters 12–13 explore connections among the Feynman path integral, Berry's phase and the Aharonov–Bohm and Aharonov–Casher effects. Quantum mechanics is nonrelativistic throughout Chaps. 1–13, but Chaps. 14–17 discuss relativistic quantum measurements, measurements of the quantum wave, and "weak" measurements within a new formalism adapted to relativity. Chapter 18 proposes simple physical axioms for quantum theory.

Our secondary goal is to encourage physicists to use paradoxes creatively, both in teaching and in research. We use paradoxes all through the book. Each chapter (except this one) begins with a paradox that motivates the rest of the chapter. Try to resolve the paradoxes as you read! Chapter 2 begins with a paradox from the class of contradictions; Chapter 10 begins with a paradox from the class of errors. Chapters 3–8 and Chaps. 11–18 all begin with paradoxes from the class of gaps; and it is an open question to which class the paradox in Chap. 9 belongs.

Problems

1.1 If Wheeler's paradox of Sect. 1.1 had preceded the discovery of quantum mechanics, to which class of paradoxes would it belong?

1.2 (a) Resolve the Triplet Paradox of Sect. 1.2. What was Jump's error?
(b) About how long did Jump's train ride last?

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2 How to Weigh a Quantum

Now we start to use paradoxes to investigate quantum theory and its mathematical formalism, quantum mechanics. We need the mathematical formalism, but we do not need it yet. We first take up the question: Is quantum theory consistent? From 1927 to 1930, Bohr and Einstein debated this question [1]. Both were familiar with the formalism, yet they hardly referred to it. They did not need to. With thought experiments, Einstein would argue that quantum theory is inconsistent; and Bohr would refute Einstein's arguments, one by one. After these refutations, Einstein conceded that quantum theory is consistent. So we can gain insight into quantum theory without the mathematical formalism. This chapter takes us to the climax of the Bohr-Einstein debate.

2.1 Why does the Color of the Light Change?

Let's visit a potter's studio. In the studio we see boxes of clay, jars of powdered glaze, spoons, brushes and rolling pins, and a potter's wheel; but what stands out is the kiln, sitting on metal posts, with its thick ceramic walls covered in metal. The huge, hot kiln dominates the studio. We peep into the hot kiln through a peephole in the door of the kiln. What do we see?

As the kiln's temperature reaches 1200 °C, the potter turns off the heating element. We then see the glow of matter heated to this temperature. The temperature of the kiln drops slowly, despite the peephole – roughly a degree per minute. So, to a good approximation, the kiln remains at thermal equilibrium. At 1200 °C (1473 K), the light we see through the peephole is orange mixed with yellow, and so bright it hurts the eyes. Almost everything in the kiln is the same color, though objects close to the peephole look darker than the background. In a few hours, the temperature of the kiln drops to 1000 °C (1273 K); the light is orange and less bright than before. We can see the outlines of some objects in the kiln. In a few more hours the temperature drops to 800 °C (1073 K) and the light is less intense; the color of the light is a mixture of orange and red. When, a few hours later, the temperature falls below 600 °C (873 K), we see only a dull red glow.

The correlation between temperature and the color of light is familiar and we take it for granted. We use such expressions as "red hot" and "white hot". A potter can judge the temperature of a kiln by the color of its glow. All the same, the correlation is mysterious. We understand why the intensity of the light changes: light has energy and the energy of the kiln decreases with the decreasing temperature. But why does the color of the light change?

Let us discuss in more detail *how* it changes. The electromagnetic radiation from a kiln is a mixture of light frequencies and other frequencies. The total energy in the radiation depends

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on the size of the kiln, but the density of energy in the radiation does not; it depends only on the temperature of the kiln. Let $u(\nu, T)$ denote the density of energy in radiation of frequency ν coming from a kiln at absolute temperature T. That is, for small $d\nu$, the density of energy in radiation with frequencies between ν and $\nu + d\nu$ is $u(\nu, T)d\nu$. In 1860, Kirchhoff proved that $u(\nu, T)$ is the same for any *black body* in thermal equilibrium at temperature T. (By definition, a black body does not reflect radiation; but it emits radiation, so, despite the name, black bodies can have color. A kiln emits black-body radiation.) But Kirchhoff did not have sufficient data to determine $u(\nu, T)$. In 1896 Wien proposed a law to fit the data that had gradually accumulated:

$$u(\nu, T) = b\nu^3 e^{-a\nu/T} ,$$

where a and b are empirical constants. Then in 1900, when new data contradicted Wien's law, Planck proposed

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} , \tag{2.1}$$

where k is Boltzmann's constant, c is the speed of light, and h is Planck's constant. (Planck's value for his constant was $h = 6.55 \times 10^{-27}$ erg sec; as of 2004, the accepted value is $h = [6.6260693 \pm 0.0000011] \times 10^{-27}$ erg sec. We note $\hbar \equiv h/2\pi \approx 1.054572 \times 10^{-27}$ erg sec.) Both Wien's law and Planck's law imply that the color of the light from a black body changes with its temperature, because the shape of $u(\nu, T)$ changes with T. And indeed the color changes, not only at temperatures that a kiln can reach, but also at higher temperatures. (See Fig. 2.1(a).)

We have defined the energy density $u(\nu, T)$ as a function of the frequency of radiation. We can just as well define energy density as a function of the wavelength λ of radiation. For small λ, let $u_\lambda(\lambda, T)d\lambda$ be the density of energy in radiation with wavelengths between λ and $\lambda + d\lambda$. For any two wavelengths $\lambda_1 = c/\nu_1$ and $\lambda_2 = c/\nu_2$, we must have

$$\int_{\nu_1}^{\nu_2} u(\nu, T)d\nu = \int_{\lambda_2}^{\lambda_1} u_\lambda(\lambda, T)d\lambda ,$$

so Eq. (2.1) implies

$$u_\lambda(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} .$$

Figure 2.1(b) shows that the light from a kiln at 1473 K is mainly red, orange, yellow and green, mixing to a yellowish orange; at 1273 K the light is mainly red, orange and yellow; at 1073 K the light is a mixture of red and orange; and at 873 K the light is dark red. (See also Prob. 2.1.) But *why* does the color of the light change?

Can we derive $u(\nu, T)$? Boltzmann and Gibbs had already invented statistical mechanics when Planck proposed his law. A principle of statistical mechanics, the equipartition theorem, states that the average kinetic energy of a system with n degrees of freedom at temperature T is $nkT/2$. Between 1900 and 1905, Rayleigh, Einstein and Jeans applied the equipartition theorem to Maxwell's electromagnetism to obtain

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} kT . \tag{2.2}$$

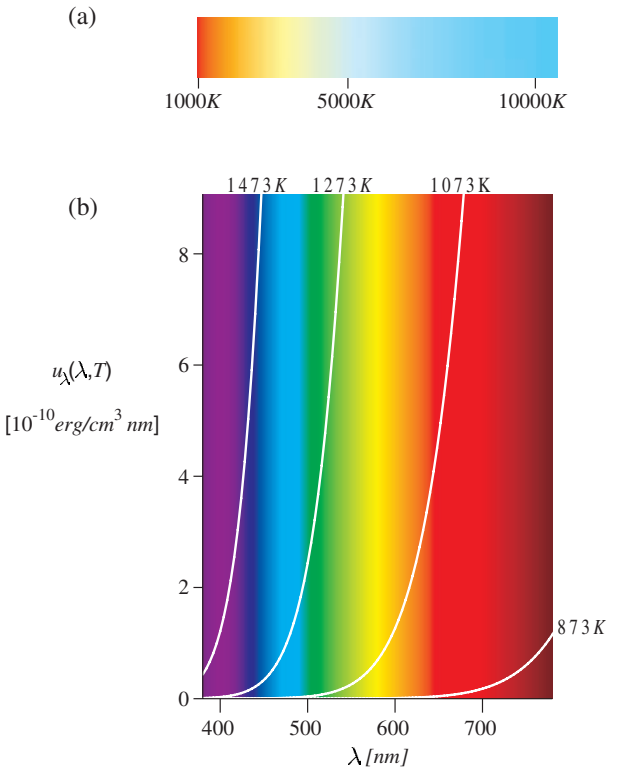

Figure 2.1: (a) Color of blackbody radiation as a function of temperature T. (b) Planck blackbody distribution $u_\lambda(\lambda, T)$ for temperatures 1473 K, 1273 K, 1073 K and 873 K, showing color of light as a function of wavelength λ. [We thank Dr. Dan Bruton for the two color spectra in this figure. Because of differences between light *emission* (from a computer screen) and *reflection* (from a printed page), the colors here differ slightly from those in his web site www.physics.sfasu.edu/astro/color.html.]

Each frequency ν of electromagnetic radiation corresponds to two degrees of freedom (two independent polarizations) and the density of frequencies is $4\pi\nu^2/c^3$. (See Prob. 2.2.) So Eq. (2.2) is simply kT times[1] the density of degrees of freedom, $8\pi\nu^2/c^3$. Note that Planck's law (unlike Wien's law) approaches Eq. (2.2) for $\nu \to 0$, but both laws contradict Eq. (2.2) for large ν.

Equation (2.2), in which T appears as an overall factor, implies that only the intensity of the light from a kiln changes with temperature, not the color. But Eq. (2.2) is wrong: if we integrate Eq. (2.2) to obtain the overall energy $U(T)$ in the radiation field,

$$U(T) = \int_0^\infty u(\nu, T) d\nu \ ,$$

[1]The factor kT includes the average of both kinetic and potential energy. These are equal for electromagnetic radiation as they are for harmonic oscillators.

we find that the integral diverges. Since statistical mechanics and electromagnetism together imply this divergence, statistical mechanics and electromagnetism together contain a contradiction. We must modify one or both of these two theories to make them compatible.

So why does the color of the light change? We can guess that it changes to avoid the divergence. The integral of Eq. (2.1) does *not* diverge. And Eq. (2.1) – Planck's law – implies that the color of the light changes with temperature. (See Prob. 2.3.)

2.2 Quanta

Planck's own derivation of Eq. (2.1) was "an act of desperation ... I had to obtain a positive result, under any circumstance and at whatever cost", as he put it [2]. Oddly, Planck was not aware of Eq. (2.2); but he was aware that he could not derive Eq. (2.1) in any reasonable way. To derive his law, he assumed that matter is composed of harmonic oscillators that exchange energy with the electromagnetic field. This assumption was reasonable enough. He also assumed that a harmonic oscillator of frequency ν could not exchange energy in arbitrary amounts, but only in *quanta* of energy $h\nu$. This assumption was completely unreasonable. According to classical theory, h should vanish; and as h vanishes, Planck's law reduces to Eq. (2.2).

Five years later, Einstein extended Planck's assumption. He assumed that electromagnetic radiation itself consists of quanta; radiation of frequency ν consists of quanta of energy E with

$$E = h\nu \ . \tag{2.3}$$

Einstein applied Eq. (2.3) to the photoelectric effect. Metals exposed to ultraviolet light emit electrons. The energy of the emitted electrons depends on the frequency, but not on the intensity, of the light. Einstein predicted a linear relation between the light frequency and the energy of the electrons, with a slope, independent of the type of metal, equal to Planck's constant. Experiments verified these predictions by 1916. Yet almost no one accepted Einstein's hypothesis of light quanta [3]. Light is a wave; how could light quanta produce interference?

Then in 1923, Compton showed that light, scattering off electrons at rest, imparts momentum in an amount that depends on the wavelength of the light, but not on its intensity. He found a clear relationship between θ, the angle through which the light scattered, and the change in its wavelength:

$$\lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta) \ ,$$

where λ_f and λ_i are the final and initial wavelengths of the light, respectively, and m is the mass of the electron. The Compton effect strongly suggests that light quanta – *photons* – of wavelength λ carry momentum h/λ as well as energy $h\nu$, for then the relationship follows from conservation of energy and momentum. (See Prob. 2.4.) In the same year, de Broglie proposed that if light waves could behave like particles, then particles could behave like waves. Four years later, Davisson and Germer observed electron diffraction and confirmed de Broglie's relation between the momentum p and the wavelength λ of a particle:

$$p = h/\lambda \ . \tag{2.4}$$

Quanta had arrived.

Section 1.4 notes that relativity theory resolves a paradox: electromagnetism and Newton's mechanics are incompatible. The theory of relativity resolves this paradox by modifying Newton's mechanics. In retrospect, we see that quantum theory, too, resolves a paradox. Statistical mechanics and electromagnetism are incompatible; together, they imply the Rayleigh–Einstein–Jeans law, Eq. (2.2), and an infinite energy density $U(T)$ for electromagnetic radiation. This is a paradox of the third class, the class of contradictions. (See Chap. 1.) Quantum theory resolves the paradox by modifying electromagnetism: electromagnetic radiation of frequency ν cannot carry energy in arbitrary amounts, but only in quanta of energy $h\nu$. Together, statistical mechanics and the modified electromagnetism imply the Planck law, Eq. (2.1), and a finite $U(T)$. (See Prob. 2.5.)

2.3 Uncertainty Relations

Quanta behave like waves and like particles. Are they waves or particles? Whatever they are, quanta confront us with a paradox each time we make a measurement. If quanta behave like waves, how can we measure their position? If they behave like particles, how can we measure their wavelength? We can live with the paradox, but it implies fundamental limits to what we measure. Here we derive these limits informally; for a formal derivation, see Prob. 3.10 and Sects. 5.3 and 7.3.

Consider a measurement with a microscope. A light microscope can resolve features of small objects, up to a limit. The limit depends on the wavelength of the light. The smallest separation Δx that a lens can resolve in its object plane (the x-axis in Fig. 2.2) is approximately

$$\Delta x \approx \lambda/2 \sin\theta \ ,$$

where λ is the wavelength of the light and θ is half the angle subtended by the lens at the object. (See Prob. 2.6.) So if we want to determine the position of a small object with an accuracy Δx, we need light of wavelength $\lambda \leq 2(\Delta x)\sin\theta$. In both classical and quantum physics, we *have* light of such short wavelengths. But in quantum physics, short wavelengths correspond to quanta carrying high momenta, as Eq. (2.4) shows. A high momentum photon scatters off the measured object and alters its momentum. Suppose we illuminate the object from the side with light of wavelength $\lambda = 2(\Delta x)\sin\theta$. The light consists of photons of momentum $p^\gamma = h/\lambda = h/2(\Delta x)\sin\theta$ in the x-direction. When a photon scatters off the object we do not know in which direction it scatters, only that it reaches the lens (if it is at all relevant to the measurement); thus all that we know about its final momentum in the x-direction is that it lies between $-p^\gamma \sin\theta$ and $p^\gamma \sin\theta$, i.e. between $-h/2\Delta x$ and $h/2\Delta x$. The photon alters the momentum of the object by this uncertain amount, hence the measurement of the object's position along the x-axis leaves us uncertain about its momentum in the x-direction; the uncertainty Δp_x in its momentum is at least $\Delta p_x \geq h/\Delta x$. In particular, we cannot rely on a prior measurement of momentum for predicting the future position of the object. This is the meaning of the Heisenberg uncertainty relation [4]:

$$\Delta x \Delta p_x \geq h \ . \tag{2.5}$$

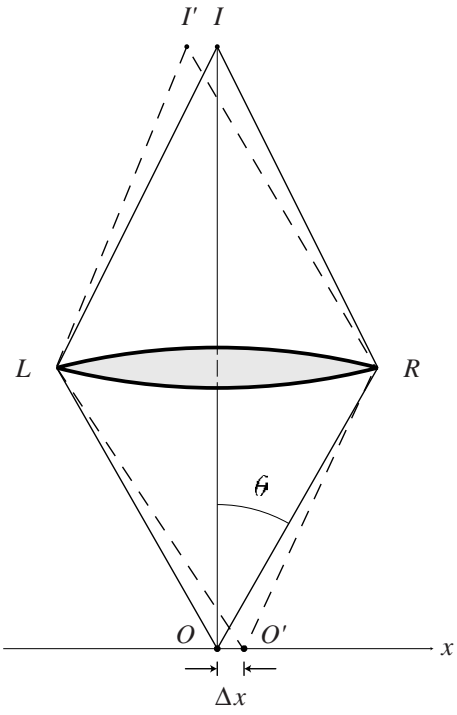

Figure 2.2: Two points O and O' with separation Δx; their respective images, each the maximum of a diffraction pattern, are I and I'.

Equation (2.5) is revolutionary. Consider, for example, an atom of hydrogen. Its radius is roughly the Bohr radius $a_0 = 0.53\ \mathring{A}$ (that is, 5.3×10^{-9} cm); its ionization energy is 13.6 electron volts (eV). Suppose we measure the position of the electron to better than the Bohr radius, i.e. $\Delta x < 5.3 \times 10^{-9}$ cm. According to Eq. (2.5), our position measurement entails uncertainty in the electron's momentum of at least

$$\Delta p > h/\Delta x \approx 1.25 \times 10^{-18}\text{g cm/sec} .$$

We might estimate the kinetic energy of the electron after the position measurement to be

$$\begin{aligned}(\Delta p/2)^2/2m &\approx (6 \times 10^{-19}\text{g cm/sec})^2/2 \times (9.1 \times 10^{-28}\text{g}) \\ &\approx 2 \times 10^{-10}\text{erg} \approx 130\text{eV} ,\end{aligned} \tag{2.6}$$

where m$\approx 9.1 \times 10^{-28}$ g is the mass of the electron. This kinetic energy is greater than the ionization energy, so the attempt to locate the electron within the atom ionizes the atom! Actually, we have overestimated the kinetic energy,[2] but any attempt to localize the electron to a well defined orbit within the hydrogen atom will indeed ionize the atom.

We obtained Eq. (2.5) from an experiment with a microscope, but Eq. (2.5) holds for any measurement of position and momentum. We always find that conditions for a precise measurement of x conflict with conditions for a precise measurement of p. The conflict illustrates

[2] A good estimate of the kinetic energy is $(\hbar/\Delta x)^2/2m \approx 14$ eV.

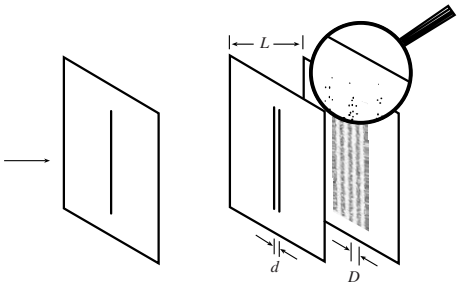

Figure 2.3: Two-slit interference experiment. Electrons enter from the left in the direction of the arrow. Magnification shows dots making up the interference pattern.

Bohr's principle of *complementarity*: measurements of canonically conjugate variables (such as x and p) impose conflicting conditions. The more an experiment fulfills the conditions for measuring one variable, the less it fulfills the conditions for measuring the conjugate variable. When we quantify the complementarity between the measurements, we obtain Eq. (2.5) (and analogous uncertainty relations for other pairs of conjugate variables).

Complementarity allows us to live with paradox, but the paradox remains. Particles and waves are complementary pictures of quanta. Each picture contains a part of the truth; but there is no picture uniting the wave and particle pictures. Indeed, they contradict each other. Consider a beam of electrons impinging on three screens. (See Fig. 2.3.) The first screen has only one slit. The second screen has two slits, separated by a distance d. The distance between the two screens is much larger than d, so waves passing through the first screen and arriving at the two slits of the second screen have effectively parallel wave vectors. The waves passing through the two slits interfere, producing a pattern of alternating light and dark bands on the third screen, a distance L from the second. The spacing between adjacent dark bands is D. This is the familiar phenomenon of wave interference, with

$$D \approx \lambda L/d \,, \tag{2.7}$$

if the wavelength is λ. What is new is that the electrons are not simply waves. They also behave like particles. If the beam intensity drops until only one electron passes through the apparatus at a time, the pattern of light and dark bands still appears. The light and dark bands emerge from marks that appear one by one on the screen, even when the time interval between successive marks is longer than the time of flight of an electron through the apparatus [5].

2.4 The Clock-in-the-Box Paradox

The double-slit experiment figured in the Bohr-Einstein debate on whether quantum theory is consistent. Einstein saw in it a paradox. Suppose we prepare the middle screen with no transverse momentum, and measure its transverse momentum after an electron passes through it. (See Fig. 2.4(a).) By measuring the recoil of the screen after the electron passes, we can infer through which slit it passed. Let us denote the electron's final transverse momentum by $p_\perp^{(L)}$ if the electron passes through the left slit and by $p_\perp^{(R)}$ if it passes through the right slit. (See Fig. 2.4(b).) If the electron passes through the left slit and arrives at point $\mathcal{P}$, the middle screen must acquire momentum $-p_\perp^{(L)}$ to conserve momentum; if it passes through the right

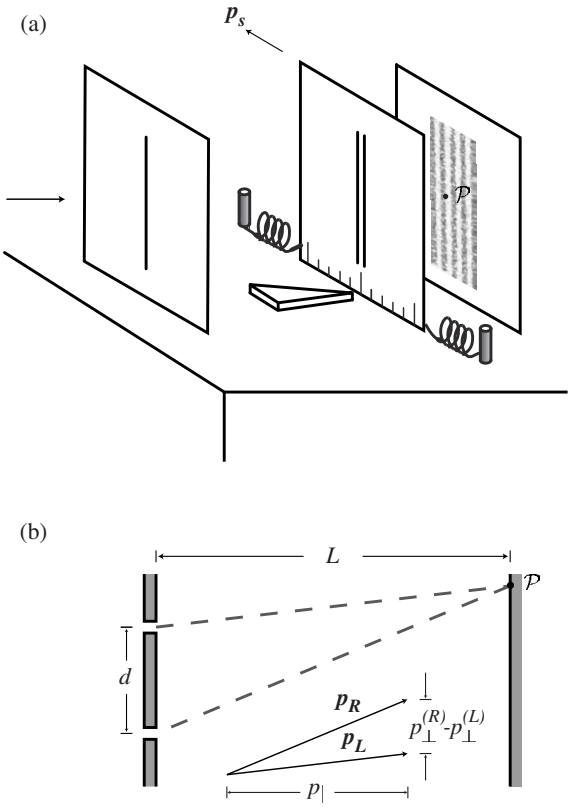

Figure 2.4: (a) The two-slit interference experiment of Fig. 2.2 adapted for measuring the transverse momentum of the middle screen. (b) The second and third screens seen from above, with interfering electron paths and corresponding momenta.

slit on its way to $\mathcal{P}$, the middle screen acquires momentum $-p_\perp^{(R)}$. Thus we can determine through which slit the particle passed by measuring the final momentum of the middle screen. How can there be an interference pattern? This is a paradox of the first class, an error. Bohr resolved the paradox by applying the uncertainty relations consistently. If we measure the momentum p_s of the screen with an accuracy Δp_s, then any simultaneous measurement of the position x_s of the screen entails an uncertainty Δx_s such that

$$\Delta x_s \geq h/\Delta p_s \ . \tag{2.8}$$

How well do we need to measure p_s? We want to detect whether a particle that arrives at $\mathcal{P}$ came via the left slit or the right one. In order to determine through which slit the electron passes, we must measure p_s to accuracy Δp_s better than $p_\perp^{(R)} - p_\perp^{(L)} = |\mathbf{p}^{(R)} - \mathbf{p}^{(L)}|$. From similarity of triangles, $p_\perp^{(R)} - p_\perp^{(L1)}$ divided by the electron's longitudinal momentum $p_\parallel$ is equal to d/L. The longitudinal momentum $p_\parallel$, according to de Broglie, is h/λ (assuming $p_\parallel$ large compared to the transverse momentum). Thus

$$\Delta p_s < \frac{d}{L}(h/\lambda) \ . \tag{2.9}$$

From Eqs. (2.7–9) we obtain $\Delta p_s < h/D$ and thus $\Delta x_s > D$. The uncertainty in the transverse *position* x_s of the screen, arising from an accurate enough measurement of its transverse *momentum* p_s, is the distance D between successive dark bands in the interference pattern. We cannot position the slits themselves more precisely than D, so the interference pattern is completely washed out. So when we *can* determine the electron path, no interference pattern remains. According to Bohr, this resolution saves us from the paradoxical conclusion that the behavior of an electron depends on the presence of a slit through which it did not pass.

Einstein posed a more subtle paradox during the sixth Solvay conference in 1930: his "clock-in-the-box paradox" seemed to violate the uncertainty relation for energy and time,

$$\Delta E \Delta t \geq h \; . \tag{2.10}$$

(See Prob. 2.8.) Imagine a box that emits a photon; a clock in the box determines the emission time. (See Fig. 2.5.) The box hangs from a spring scale which measures its weight. Its weight is proportional to its rest mass m, hence to its energy E, according to Einstein's formula $E = mc^2$. Einstein argued as follows: Suppose we wait for the box to settle down and make an accurate initial reading of the scale. This reading can be as slow and accurate as we like. After the photon leaves the box, we make a final reading of the scale, again as accurate as we like, and from the difference between the two readings we get an accurate measurement of the energy of the emitted photon. On the other hand, the clock in the box tells exactly when the photon was released, so we violate Eq. (2.10). Rosenfeld, who was soon to become Bohr's assistant, described Bohr's reaction to this argument: "It was quite a shock for Bohr to be faced with this problem; he did not see the solution at once. During the whole evening he was extremely unhappy, going from one to the other and trying to persuade them that it couldn't be true, that it would be the end of physics if Einstein were right; but he couldn't produce any refutation ... The next morning came Bohr's triumph and the salvation of physics ... ." [6].

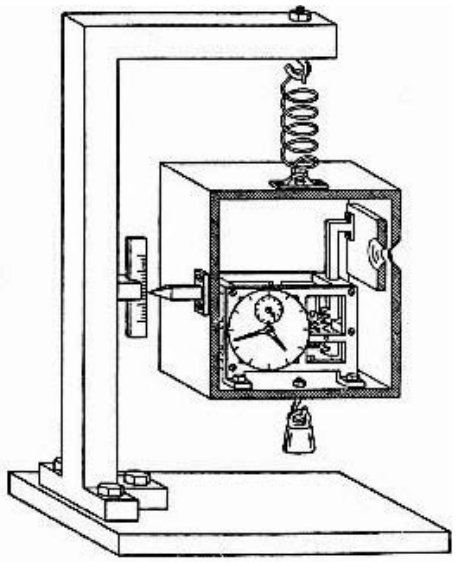

Figure 2.5: Bohr's setup for Einstein's clock-in-the-box paradox. [Reprinted by permission of Open Court Publishing Company, a division of Carus Publishing Company, Peru, IL, from *Albert Einstein: Philosopher–Scientist* (Library of Living Philosophers **VII**), ed. P. A. Schilpp, copyright © 1949 and 1951 by the Library of Living Philosophers, Inc.]

Bohr triumphed as follows. Let x denote the position of the pointer on the scale and p its momentum, with Δx and Δp the corresponding uncertainties. Once we choose Δx, Heisenberg's uncertainty relation bounds Δp from below:

$$\frac{h}{\Delta x} \leq \Delta p \ . \tag{2.11}$$

Bohr assumed a standard procedure for weighing the photon: After the photon emission, the pointer moves (higher on the scale). By hanging little weights on the box, we lower it to its original position. When it has returned to its original height, the total weight hanging from it equals the weight of the emitted photon. However, the accuracy of this weighing is no better than the smallest added weight $g\Delta m$ that has an observable effect. If we add a mass Δm and wait a time t, the impulse delivered to the box cannot be greater than $(g\Delta m)t$, which must be greater than Δp to be observable. Thus we have

$$\Delta p < gt\Delta m \ . \tag{2.12}$$

From Eqs. (2.11–12) we obtain $h \leq \Delta x \Delta p < gt\Delta x \Delta m$. Using $\Delta E = c^2 \Delta m$ (because we infer the energy of the photon from the change in mass of the box) we obtain

$$h < \frac{gt\Delta x \Delta E}{c^2} \ . \tag{2.13}$$

Still, Eq. (2.13) does not resolve Einstein's paradox, for ΔE is paired with Δx instead of Δt.

Einstein assumed that the reading of the pointer could take unlimited time. But Bohr applied a result from Einstein's own theory of general relativity. According to the time-dilation formula of general relativity, a clock in a gravitational field ticks more slowly than a clock in free fall. Two clocks at different heights above the Earth will run at different rates, because of their gravitational potential difference. If the difference in height is Δx, the fractional difference $\Delta t/t$ in their measured times will be

$$\frac{\Delta t}{t} = \frac{g\Delta x}{c^2} \ . \tag{2.14}$$

We can also let Δx and Δt in Eq. (2.14) denote uncertainties; if Δx is the uncertainty in the vertical position of a clock, then Δt is the uncertainty in the clock time due to the uncertain gravitational potential. Over a period t, the uncertainty in the time of the clock amounts to

$$\Delta t = tg\Delta x/c^2 \ .$$

Combining this result with Eq. (2.13) we obtain

$$\Delta t \Delta E > h \ ,$$

as required by quantum theory. Bohr's triumph is impressive – and also, at first glance, spooky. How does quantum theory know about general relativity? The answer is simple. To measure the mass of the box and its contents, we weigh it. But energy is equivalent to *inertial* mass. We weigh the *gravitational* mass. By assuming them to be equal, we have assumed the principle of equivalence, one of the axioms of general relativity. This axiom and the equivalence of mass and energy imply the time-dilation relation of general relativity, Eq. (2.14). (See Prob. 2.9.)

2.5 From Inconsistency to Incompleteness

The clock-in-the-box paradox was a turning point in Einstein's criticism of quantum theory. Before, he invested his efforts in proving that quantum theory is inconsistent. These efforts failed; the clock-in-the-box paradox, too, belongs to the class of errors. After the Solvay conference, Einstein stopped trying to prove the inconsistency of quantum theory, and instead tried to prove that quantum mechanics is incomplete. Einstein's efforts to prove the incompleteness of quantum theory led him to a paradox, the paradox of Einstein, Podolsky and Rosen (EPR), which opens the next chapter. Unlike the clock-in-the-box paradox, the EPR paradox was not resolved overnight; it has taken decades to unravel.

Problems

2.1 Consider two frequencies of light, $\nu_R = 4.3 \times 10^{14}$/sec (red) and $\nu_Y = 5.2 \times 10^{14}$/sec (yellow).
(a) For radiation from a black body at temperatures T equal to 1473 K, 1273 K, 1073 K and 873 K, compute the ratio $u(\nu_Y, T)/u(\nu_R, T)$ according to the Planck law, Eq. (2.1).
(b) Apply these results to estimate the color of black body radiation at these temperatures, assuming that the human eye is six times more sensitive to yellow light than to red light.
(c) Compute the frequencies that maximize $u(\nu, T)$ at these temperatures.

2.2 (a) Show that in a cube of volume L^3 with periodic boundary conditions, wave numbers $\mathbf{k}$ satisfy $k_x = 2\pi n_x/L$, $k_y = 2\pi n_y/L$ and $k_z = 2\pi n_z/L$, where n_x, n_y and n_z are integers.
(b) Derive the density of states

$$\frac{dn_x dn_y dn_z}{L^3} = \frac{dk_x dk_y dk_z}{(2\pi)^3} \ .$$

(c) Derive

$$\frac{dk_x dk_y dk_z}{(2\pi)^3} = \frac{4\pi\nu^2}{c^3} d\nu$$

in the case of spherical symmetry.

*2.3 Suppose that the energy density $u(\nu, T)$ of radiation from a black body is unknown, but known to approach Eq. (2.2) as ν approaches 0. Show that the integral $\int_0^\infty u(\nu, T) d\nu$ diverges unless the color of light from a black body changes with its temperature, i.e. unless $u(\nu, T)$ does not scale with T. (Note that there is no fundamental constant of length in electromagnetism and statistical mechanics.)

2.4 Apply relativistic energy and momentum conservation to the scattering of massless photons off electrons at rest to derive Compton's relation $\lambda_f - \lambda_i = (h/mc)(1 - \cos\theta)$, where λ_i and λ_f are the initial and final photon wavelengths, m is the electron mass and θ is the angle of scattering of the photons.

2.5 Derive the Planck law, Eq. (2.1), from the partition function for the degrees of freedom of electromagnetic radiation at temperature T, assuming Eq. (2.3) and treating each mode as a harmonic oscillator with energy levels $h\nu, 2h\nu, 3h\nu, \ldots$.

2.6 Diffraction limits the resolving power of a lens. According to *Rayleigh's criterion*, a lens barely resolves two points O, O' if the central maximum of the diffraction pattern of O lies within the first minimum of the diffraction pattern of O'. (See Fig. 2.2.) For side illumination with light of wavelength λ, Rayleigh's criterion implies [7] a difference between the extremal paths $O'LI$ and $O'RI$ of about one wavelength λ. Show that $\Delta x \approx \lambda/2 \sin\theta$ follows from this criterion.

2.7 Show how the microscope measurement in Sect. 2.3 illustrates the principle of complementarity for position vs. momentum.

2.8 Consider a free particle of energy E and momentum p passing a point on its trajectory at a time t. Assume $p \gg \Delta p$.
(a) Express the uncertainties ΔE and Δt in terms of p, Δp and the uncertainty Δx in the particle's position.
(b) Show that $\Delta x \Delta p \geq h$ implies $\Delta E \Delta t \geq h$.

2.9 Consider a rocket with an upper and a lower chamber. A clock in the upper chamber emits a flash every T seconds. A detector in the lower chamber registers each flash. The distance between the clock and the detector is Δx (in the rest frame of the rocket) and the rocket accelerates with constant acceleration g.
(a) Consider an inertial reference frame moving with the same velocity $\mathbf{v}$ as the rocket at a given instant. Assume $v \ll c$ and show that in this inertial reference frame the flash reaches the detector after a time $t = \Delta x/(c+v)$.
(b) Show that the time interval between flashes at the detector is $T - \Delta T$ where

$$\frac{\Delta T}{T} = \frac{g \Delta x}{c^2} \, .$$

(c) Obtain Eq. (2.14) from the principle of equivalence.

*2.10 Consider a frictionless track that is mostly horizontal but inclined at an angle θ at both ends, as in Fig. 2.6. We release a box from one end. Before it reaches the other end, the box emits a photon at a time t determined by a clock in the box. If m is the mass of the box and its contents, then the energy E of the photon equals the change in mc^2. Suppose we measure p, the momentum of the box on the horizontal part of the track, and y, its height at rest on the inclined part, before and after the emission. Then we can determine m by equating the kinetic energy $p^2/2m$ and potential energy mgy of the box:

$$m = p/(2gy)^{1/2}$$

Show how the uncertainty relation $\Delta E \Delta t \geq h$ applies to this experiment.

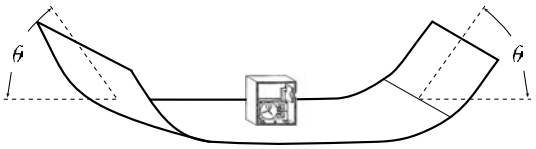

Figure 2.6: The clock in the box of Fig. 2.4 on a mostly horizontal, frictionless track with ends inclined at an angle θ. [Adapted from Fig. 2.5 by permission of Open Court Publishing Company.]

*2.11 A box of mass M emits a photon at time t according to a clock in the box. The box moves horizontally on a frictionless track; it is attached to a spring with spring constant k, known to great precision. Before and after the photon emission, we measure the angular frequency ω of the harmonic motion to great precision (by measuring the time it takes the box to complete many periods). We calculate the energy E of the emitted photon from the change in mc^2 where $m = k/\omega^2$. Show that $\Delta E \Delta t \geq h$, where ΔE and Δt are the uncertainties in the energy and emission time of the photon.

*2.12 A heavy box of mass M emits a photon at a time t according to a clock in the box. Assume that the box is initially at rest. We measure its initial mass as follows: We hit it with a particle of known mass m and initial momentum p_0, and measure the final momenta p, P of the particle and of the box, respectively; from conservation of momentum and energy we derive M. We then stop the box's motion by hitting it a second time with the same particle, but with momentum $-p$. After the emission of the photon, we measure the final mass of the box in the same way.
(a) Express the mass M in terms of m, p, p_0 and P.
(b) Calculate the uncertainty ΔE in the energy of the photon and show that $\Delta E \Delta t \geq h$.

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3 Is Quantum Theory Complete?

In quantum theory, every measurement entails a choice. If we choose to measure one observable (physical variable), we choose *not* to measure some other observable. In confronting us with these choices quantum theory is consistent, but does quantum theory provide a complete account of all that we observe? Is quantum theory perhaps incomplete?

The title of a paper by Einstein, Podolsky and Rosen [1] (EPR) asks this question. So does the title of a reply by Bohr [2]. Einstein, Podolsky and Rosen considered the possibility of a complete theory. Such a theory, they claimed, must represent *both* the position and the momentum of a particle at any time, with arbitrary accuracy. Quantum theory does not; hence, quantum theory is incomplete.

The next section presents the EPR claim that quantum theory is incomplete. Einstein, Podolsky and Rosen believed a complete theory to be possible, although they did not show how to complete the theory. *Is* there such a complete theory? This question has a remarkable answer – one which Einstein, Podolsky, Rosen and Bohr did not anticipate.

3.1 The Einstein–Podolsky–Rosen Paradox

Einstein, Podolsky and Rosen used quantum mechanics in their paper. But the mathematical formalism is not necessary. Here we present the EPR paradox without the formalism, just as Bohr did in his reply to the EPR paper.

Suppose two particles that once interacted are now apart and no longer interact. Since they do not interact, a measurement on one particle does not affect the other particle. But since they interacted in the past, a measurement on one particle may be an indirect measurement on the other particle. For example, we can measure the position of a particle indirectly. In Fig. 3.1, a pair of massive particles approaches a board with two slots in it, a distance L apart. Suppose that the particles pass through the board through different slots, at the same time. (It may be unlikely that they pass through the two slots, but we can try many times until two particles do pass through the two slots simultaneously.) In this way we establish that the vertical distance between two particles, as they pass through the board, is L. Now we measure the height of one of the particles right after they pass through the board. We indirectly measure the height of the other particle, too, without affecting it.

We can combine the position measurement with a momentum measurement. Consider measurements of position and momentum along the vertical axis (x-axis) in Fig. 3.2. Particles arrive at the board in pairs, as in Fig. 3.1; let x_1 and p_1 denote the position and momentum components of one particle, and x_2 and p_2 the position and momentum components of the

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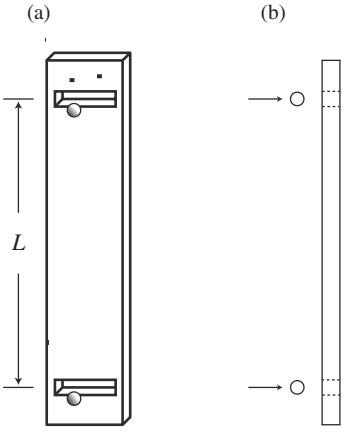

Figure 3.1: (a) An experiment to measure the height of one particle relative to another. (b) Side view of the experiment.

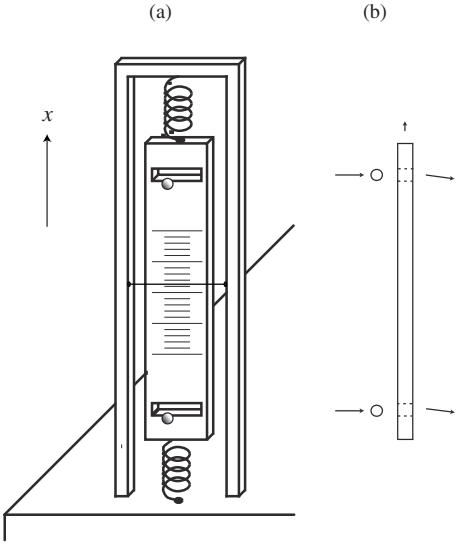

Figure 3.2: (a) An experiment to measure $x_2 - x_1$, the relative height of two particles, and $p_1 + p_2$, their total momentum along the vertical axis, at the same time. (b) Side view of the experiment. Initially, $p_1 = 0 = p_2$; while p_1 and p_2 may change as the particles pass through the slots, the vertical recoil of the board measures the change in $p_1 + p_2$.

other. We arrange that initially, $p_1 = p_2 = 0$; the initial momenta are horizontal. Also, before and after each pair arrives at the board, we accurately measure the momentum component of the board itself. Since the total momentum of the particles and the board remains constant during the passage, this experiment yields the total momentum $p_1 + p_2$ of the particles after they pass through the board (even if they knock against the slots as they pass through). It also yields the vertical distance $x_2 - x_1$ between the particles as they pass through the board (if they pass through the two slots). Right after they pass through the board, we can indirectly measure either the position or momentum of one particle in a pair without affecting it. For example, from a direct measurement of x_1 we can infer x_2 and from a direct measurement of p_1 we can infer p_2.

The measurements in this thought experiment do not contradict the uncertainty relations.[1] We measure the momentum of the board, but we never have to measure the position of the board. We prepare pairs of particles for which p_1 and p_2 vanish initially, but we assume nothing about x_1 and x_2 initially. Bohr showed, however, that the EPR experiment confronts us with a choice between complementary measurements. We can measure x_1 and infer x_2 from the result. However, measuring x_1 may affect p_1. After measuring x_1, we cannot measure p_1 and infer p_2 from the result. Alternatively, we can measure p_1 and infer p_2 from the result. However, measuring p_1 may affect x_1; after measuring p_1, we cannot measure x_1 and infer x_2 from the result. We could also measure p_2 indirectly and x_2 directly. But the x_2 measurement affects p_2 just as it does when we measure p_2 and x_2 directly. Thus Δx_2 and Δp_2 satisfy the same uncertainty relation whether we measure x_2 or p_2 directly or indirectly.

So once again, Bohr showed that quantum theory is consistent. But is quantum theory complete? Einstein, Podolsky and Rosen did not claim that quantum theory is, or is not, consistent; they claimed that quantum theory is not complete. Let us consider the EPR claim.

We return to the thought experiment. Suppose we prepare a pair of particles with $x_2 - x_1 = L$. If the board does not recoil up or down, then we also have $p_1 + p_2 = 0$. Next we measure p_2 and either x_1 or p_1. By assumption, the particles do not interact, so the result of the p_2 measurement cannot depend on whether we measure x_1 or p_1. What is p_2? If we measure p_1 and obtain $p_1 = p$, we also obtain $p_2 = -p$ because $p_1 + p_2 = 0$. If we measure x_1 we cannot measure p_1, and quantum theory does not predict the result of the p_2 measurement. But the result of the p_2 measurement cannot depend on whether we measure x_1 or p_1, by assumption. Hence we must still obtain $p_2 = -p$, i.e. some predetermined result independent of what we measure on the other particle (and which particle we measure first). Since quantum theory does not predict this result, quantum theory is incomplete. This is the EPR claim.

The EPR claim is a paradox for quantum uncertainty. Let two observers, Alice and Bob, prepare a pair of particles with $x_2 - x_1 = L$ and $p_1 + p_2 = 0$ where L is very large. Next Alice, located near one particle, measures either x_1 or p_1; Bob, located near the other particle, measures p_2. They choose independently what to measure and their measurements are spacelike separated. Later they may meet and compare their results, but the experiment is over once they complete their measurements. In some inertial reference frame, Alice measures first. She measures either the position or the momentum of her particle. If Alice measures momentum, Bob's uncertainty about the momentum of his particle (before he measures it) is subjective, according to Alice. If Alice measures position, Bob's uncertainty about the momentum of his particle (before he measures it) is objective, according to the uncertainty relations. But in another frame, Bob measures first, and we cannot distinguish subjective from objective uncertainty.

Does the EPR claim have implications for experiment? We might well wonder. Pauli, writing to Born in 1954, expressed doubt: "As O. Stern said recently, one should no more rack one's brain about the problem of whether something one cannot know anything about exists all the same, than about the ancient question of how many angels are able to sit on the point of a needle. But it seems to me that Einstein's questions are ultimately always of this kind" [3].

[1] Formally, we have $[x_2 - x_1, p_1 + p_2] = 0$, so the uncertainty relations do not constrain simultaneous measurements of $x_2 - x_1$ and $p_1 + p_2$. (See Prob. 3.10.)

Yet Pauli was wrong; the EPR claim is testable! Ten years later, Bell discovered that the EPR claim implicitly contradicts predictions of quantum mechanics.

3.2 Polarized Photons

If we replace the position and momentum measurements in the EPR thought experiment with spin or polarization measurements, it becomes a practical experiment. Bohm reformulated the EPR paradox for such an experiment [4]. At one stroke he confronted the EPR claim with experiment and with quantum theory. In this section we discuss polarization in classical and quantum theory; in Sect. 3.4 we present Bohm's formulation and apply it – as Bell did – to show that the EPR claim contradicts predictions of quantum mechanics.

Figure 3.3 shows two polarizers – disks of polaroid or another polarizing material – with a common axis of rotation. Light from a lamp or the sun shines on the first polarizer, through a pinhole on the axis; some of this light passes through the polarizers. What happens if we rotate one polarizer with respect to the other? Experiment shows a change in the intensity of light passing through the polarizers. Each polarizer has a *pass axis*, and the second polarizer passes a fraction $\cos^2(\theta-\theta')$ of the light that passes the first polarizer, where $\theta-\theta'$ is the angle between their pass axes. What accounts for this effect? Let us first consider the classical, and then the quantum, account.

In classical theory, light is an electromagnetic wave. Like any electromagnetic wave, light is either monochromatic or a superposition of monochromatic waves; a monochromatic wave has a propagation vector $\mathbf{k}$ (which fixes the angular frequency $\omega = c|\mathbf{k}|$ of the wave) and a polarization (which fixes the direction of the electric field $\mathbf{E}(\mathbf{x},t)$). Since there are two independent directions for $\mathbf{E}$ – which must be perpendicular to $\mathbf{k}$ – there are two independent polarizations. Let unit vectors $\epsilon_{\mathbf{1}}$ and $\epsilon_{\mathbf{2}}$, with $\epsilon_{\mathbf{1}}\cdot\epsilon_{\mathbf{2}} = \epsilon_{\mathbf{1}}\cdot\mathbf{k} = \epsilon_{\mathbf{2}}\cdot\mathbf{k} = 0$, represent the two (linear) polarizations. Then the electric field of a monochromatic wave is a superposition

$$\mathbf{E}(\mathbf{x},t) = E_1\epsilon_{\mathbf{1}}\cos(\mathbf{k}\cdot\mathbf{x}-\omega t+\phi_1) + E_2\epsilon_{\mathbf{2}}\cos(\mathbf{k}\cdot\mathbf{x}-\omega t+\phi_2) \tag{3.1}$$

with real phases ϕ_1, ϕ_2 and coefficients E_1, E_2. Without loss of generality we can take $|\phi_2-\phi_1| \le \pi/2$.

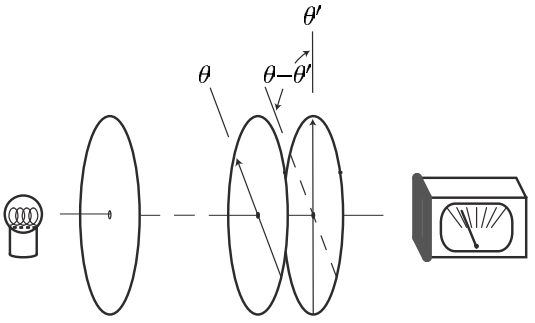

Figure 3.3: Two polarizers with an angle $\theta-\theta'$ between their pass axes.

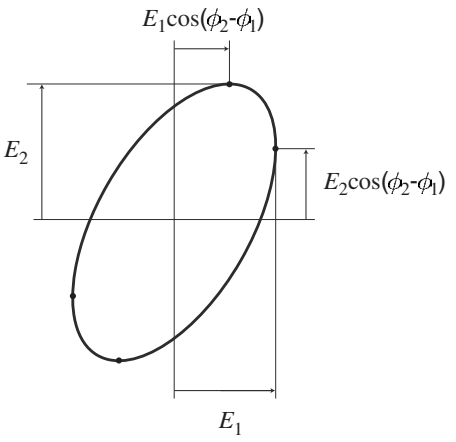

Figure 3.4: The electric field $\mathbf{E}(\mathbf{x}, t)$ of Eqs. (3.1–2) traces out an ellipse at each point $\mathbf{x}$.

If $\phi_1 = \phi_2$, the polarization is linear. Otherwise, the polarization is elliptic. A special case of elliptic polarization is circular polarization: $|E_1| = |E_2|$ and $\phi_1 - \phi_2 = \pm\pi/2$. (See Fig. 3.4.) Now a polarizer passes only the component of $\mathbf{E}$ that is parallel to its pass axis; hence light that passes through a polarizer is linearly polarized. Let us represent the polarization of the light passing through the first polarizer as a unit vector $\mathbf{P} = P_1\boldsymbol{\epsilon_1} + P_2\boldsymbol{\epsilon_2} = \cos\theta\,\boldsymbol{\epsilon_1} + \sin\theta\,\boldsymbol{\epsilon_2}$ parallel[2] to $\mathbf{E}$. Of this light, only the component parallel to $\mathbf{P'} = P'_1\boldsymbol{\epsilon_1} + P'_2\boldsymbol{\epsilon_2} = \cos\theta'\boldsymbol{\epsilon_1} + \sin\theta'\boldsymbol{\epsilon_2}$ passes through the second polarizer. So the second polarizer reduces the amplitude of $\mathbf{E}$ by $\mathbf{P}\cdot\mathbf{P'} = P_1P'_1 + P_2P'_2 = \cos(\theta - \theta')$ and reduces the intensity (which is proportional to the amplitude squared) by $\cos^2(\theta - \theta')$. The experiment in Fig. 3.3 shows the effect of linear polarization.

Now what happens if we reduce the incident light intensity? Classical theory predicts, and experiment confirms, that the same fraction of light, $\cos^2(\theta - \theta')$, passes the second polarizer, whatever the incident light intensity. But what if we reduce the incident light intensity until the energy of the light passing through the polarizers is roughly $\hbar\omega = h\nu$? According to quantum theory, the incident light consists of photons of energy $\hbar\omega$. (See Eq. (2.3).) Hence only one photon passes through a polarizer at a time. This photon either does or does not pass the second polarizer. It cannot split into a part that passes the polarizer and a part that does not, because its angular frequency ω, and therefore its energy $\hbar\omega$, cannot change as it passes through the polarizer. So $\cos^2(\theta - \theta')$ must be the *probability* that a photon arriving at the second polarizer will pass through it.

We can generalize this experiment by using birefringent panes [5]. It is convenient to express $\mathbf{E}(\mathbf{x}, t)$ as the real part of a complex wave [6]:

$$\mathbf{E}(\mathbf{x}, t) = \Re\ \left[\ (E_1 e^{i\phi_1}\boldsymbol{\epsilon_1} + E_2 e^{i\phi_2}\boldsymbol{\epsilon_2})e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}\ \right]\ . \tag{3.2}$$

Because electric fields add as vectors, the polarizations $E_1 e^{i\phi_1}\boldsymbol{\epsilon_1}$ and $E_2 e^{i\phi_2}\boldsymbol{\epsilon_2}$ add as vectors, even when they are complex. Equation (3.2) allows us to represent the polarization of an

[2] If necessary, we can change the sign of $\mathbf{E}$ by adding π to ϕ_1 and to ϕ_2.

electromagnetic wave as a vector $|\mathbf{P}\rangle$ in a two-dimensional complex vector space:

$$|\mathbf{P}\rangle = \frac{E_1}{E} e^{i\phi_1} |\epsilon_\mathbf{1}\rangle + \frac{E_2}{E} e^{i\phi_2} |\epsilon_\mathbf{2}\rangle \ ,$$

where $E = (E_1^2 + E_2^2)^{1/2}$. We let $|\mathbf{P}\rangle$, $|\epsilon_\mathbf{1}\rangle$, $|\epsilon_\mathbf{2}\rangle$ denote vectors in the complex vector space of polarizations, while $\mathbf{P}$, $\epsilon_\mathbf{1}$, $\epsilon_\mathbf{2}$, denote vectors in the real vector space of directions perpendicular to $\mathbf{k}$. Now a birefringent pane has an *optic axis*, and it shifts the relative phase between polarization components (i.e. between the component parallel to the optic axis and the component perpendicular to it) by an amount that depends on the thickness of the pane. By placing birefringent panes on either side of a polarizer, we can make it select (pass) monochromatic light with arbitrary polarization. We have a selector for arbitrary polarizations, not just linear polarizations. Figure 3.5 shows two such selectors. Suppose the first selector selects light of polarization $|\mathbf{P}\rangle$ and the second selector selects light of polarization $|\mathbf{P}'\rangle$. What fraction of light passing the first selector also passes the second selector? Prob. 3.2 shows that the fraction is

$$|\langle \mathbf{P}|\mathbf{P}'\rangle|^2 \ , \tag{3.3}$$

where the scalar product $\langle \mathbf{P}|\mathbf{P}'\rangle$ between two vectors $|\mathbf{P}\rangle = P_1|\epsilon_\mathbf{1}\rangle + P_2|\epsilon_\mathbf{2}\rangle$ and $|\mathbf{P}'\rangle = P_1'|\epsilon_\mathbf{1}\rangle + P_2'|\epsilon_\mathbf{2}\rangle$ generalizes $\mathbf{P}\cdot\mathbf{P}'$:

$$\langle \mathbf{P}|\mathbf{P}'\rangle = P_1^* P_1' + P_2^* P_2' \ .$$

(The asterisk denotes complex conjugation.) For example, if $|\mathbf{P}\rangle = |\epsilon_\mathbf{2}\rangle$ and $|\mathbf{P}'\rangle = (|\epsilon_\mathbf{1}\rangle + i|\epsilon_\mathbf{2}\rangle)/\sqrt{2}$, then $\langle \mathbf{P}|\mathbf{P}'\rangle = i/\sqrt{2}$ and the fraction is 1/2. If $|\mathbf{P}\rangle = |\mathbf{P}'\rangle$, the photon will certainly pass through the selector. If $|\mathbf{P}\rangle$ is orthogonal to $|\mathbf{P}'\rangle$, i.e. $\langle \mathbf{P}|\mathbf{P}'\rangle = 0$, the photon will certainly not pass. In general, $|\mathbf{P}\rangle$ is a complex superposition of $|\mathbf{P}'\rangle$ and the orthogonal polarization, and a fraction $|\langle \mathbf{P}|\mathbf{P}'\rangle|^2$ of the photons that pass through the first selector also pass through the second selector. When the polarizations $|\mathbf{P}\rangle$ and $|\mathbf{P}'\rangle$ are linear, Eq. (3.3) reduces to $\cos^2(\theta - \theta')$.

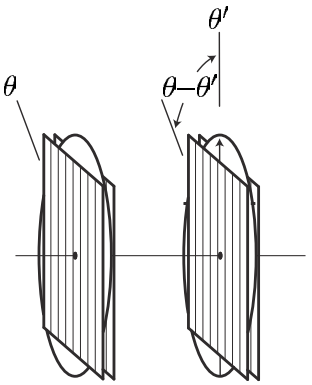

Figure 3.5: Each polarizer, sandwiched between birefringent panes, selects a particular elliptical polarization. Assume the two birefringent panes have optic axes parallel to $\epsilon_\mathbf{1}$. The first birefringent pane has just the right thickness to convert a particular elliptical polarization to the linear polarization that this polarizer passes; the second birefringent pane then converts the linear polarization back to the original elliptical polarization by restoring the relative phase $\phi_2 - \phi_1$.

3.3 Quantum States and Observables

Photon polarization illustrates a correspondence between quantum states and measurements. The polarization $|P\rangle$ is an example of a quantum state in a two-dimensional complex vector space – Hilbert space. The Hilbert space is two-dimensional, because if a photon arrives at a polarizer, there are two possible outcomes: the photon either does or does not pass through the polarizer. A measurement may have more than two possible outcomes. Section 3.4 considers measurements on *pairs* of photons. Each photon in a pair arrives at a polarizer and either does or does not pass through. There are four mutually exclusive answers to the question, "Do the two photons pass through the polarizers?" Thus polarization states of two photons are vectors in a four-dimensional Hilbert space; a basis for this space is $|\epsilon_{\mathbf{1}}\rangle \otimes |\epsilon_{\mathbf{1}}\rangle$, $|\epsilon_{\mathbf{2}}\rangle \otimes |\epsilon_{\mathbf{1}}\rangle$, $|\epsilon_{\mathbf{1}}\rangle \otimes |\epsilon_{\mathbf{2}}\rangle$ and $|\epsilon_{\mathbf{2}}\rangle \otimes |\epsilon_{\mathbf{2}}\rangle$, where the first state in each product represents one photon and the second state represents the other. The Hilbert space of a quantum system has dimension N if a complete measurement on the system has N possible outcomes.

This correspondence between quantum states and measurements follows the principle of complementarity. There are only two basis vectors for the polarization state of a photon, because an experiment can select only one (and reject only one) polarization at a time. There are infinitely many basis vectors for the position of a particle, because an experiment that selects one position rejects infinitely many other positions. (See Prob. 3.9.) But there are no additional basis vectors for the momentum of the particle, because the principle of complementarity allows us to measure *either* the position *or* the momentum of a particle accurately. There *are* basis vectors for momentum, but they are superpositions of the basis vectors for position (and vice versa). Incompatible measurements on a system correspond to incompatible bases for the Hilbert space of the system.

By measuring an observable A on a system, we choose a basis of states. Let us denote the basis states $|i\rangle$, where $i = 1, \ldots, N$ for an N-dimensional Hilbert space, and take $\langle i|j\rangle = \delta_{ij}$ – the basis is orthonormal. (See Prob. 3.5.) The correspondence with A is that if the system is in a state $|i\rangle$, a measurement of A yields an outcome a_i with certainty. [3] All other possible states $|\psi\rangle$ of the system are linear combinations of the $|i\rangle$ with complex coefficients c_i,

$$|\psi\rangle = \sum_i c_i |i\rangle \ ,$$

with the constraint $\sum_i c_i^* c_i = 1$, just as any polarization vector is a linear combination of $|\epsilon_{\mathbf{1}}\rangle$ and $|\epsilon_{\mathbf{2}}\rangle$ with complex coefficients satisfying the same constraint. The scalar product of two states $|\psi\rangle$ and $|\psi'\rangle$, denoted $\langle\psi|\psi'\rangle$, is the generalization of $\langle \mathbf{P}|\mathbf{P}'\rangle$ to Hilbert spaces of more than two dimensions. That is, if $|\psi'\rangle$ is

$$|\psi'\rangle = \sum_i c_i' |i\rangle \ ,$$

then the scalar (or inner) product of $|\psi\rangle$ and $|\psi'\rangle$ is

$$\langle\psi|\psi'\rangle = \sum_i c_i^* c_i' \ .$$

[3] If A is degenerate, i.e. if $a_i = a_j$ for some $i \neq j$, we also measure a nondegenerate observable on $|i\rangle$ and $|j\rangle$ for a complete measurement.

The square of the absolute value of $\langle\psi|\psi'\rangle$ is the probability that a measurement on a system in the state $|\psi\rangle$ will show it to be in the state $|\psi'\rangle$ (or vice versa), just as $|\langle\mathbf{P}|\mathbf{P}'\rangle|^2$ represents the probability that a measurement on a photon with polarization $|\mathbf{P}\rangle$ will show it to have polarization $|\mathbf{P}'\rangle$ (or vice versa).

Now consider an experiment in which we prepare a system in a state $|\psi\rangle$ and measure A. With probability $|\langle\psi|i\rangle|^2$ the measurement leaves the system in a state $|i\rangle$ and the measured value of A is a_i. (The constraint $\sum_i c_i^* c_i = 1$ insures that the probabilities $|\langle\psi|i\rangle|^2$ sum to 1.) If we repeat this experiment (the preparation and the measurement) many times, the average measured value of A will be

$$\sum_i a_i |\langle\psi|i\rangle|^2 \ ,$$

which we can write $\langle\psi|A|\psi\rangle$ by defining a linear Hermitian operator A:

$$A = \sum_i a_i |i\rangle\langle i| \ . \tag{3.4}$$

All observables of quantum theory are linear Hermitian operators. (See Probs. 3.3–4.) We call $\langle\psi|A|\psi\rangle$ the expectation value of A in the state $|\psi\rangle$, letting A denote both the physical variable and its operator. Applying A to the state $|j\rangle$ we find

$$A|j\rangle = a_j|j\rangle;$$

$|j\rangle$ is an eigenstate (or eigenvector) of A with eigenvalue a_j.

3.4 Bell's Inequality

A pion at rest decays into two photons. The photons fly off in opposite directions, with equal and opposite momenta. What is their polarization state? It must be a linear combination of the four basis vectors $|\epsilon_\mathbf{1}\rangle \otimes |\epsilon_\mathbf{1}\rangle$, $|\epsilon_\mathbf{1}\rangle \otimes |\epsilon_\mathbf{2}\rangle$, $|\epsilon_\mathbf{2}\rangle \otimes |\epsilon_\mathbf{1}\rangle$ and $|\epsilon_\mathbf{2}\rangle \otimes |\epsilon_\mathbf{2}\rangle$, but which linear combination? The same question arises for the annihilation of a positron and an electron into two photons in the decay of positronium, and for the emission of photon pairs in an atomic cascade. Experiment shows that the polarization state of photons from pion or positronium decay is

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}} \left[|\epsilon_\mathbf{1}\rangle \otimes |\epsilon_\mathbf{2}\rangle - |\epsilon_\mathbf{2}\rangle \otimes |\epsilon_\mathbf{1}\rangle \right] \tag{3.5}$$

and not one of the states $|\Psi_2\rangle$, $|\Psi_3\rangle$ and $|\Psi_4\rangle$:

$$\begin{aligned}
|\Psi_2\rangle &= \frac{1}{\sqrt{2}} \left[|\epsilon_\mathbf{1}\rangle \otimes |\epsilon_\mathbf{2}\rangle + |\epsilon_\mathbf{2}\rangle \otimes |\epsilon_\mathbf{1}\rangle \right] \ , \\
|\Psi_3\rangle &= \frac{1}{\sqrt{2}} \left[|\epsilon_\mathbf{1}\rangle \otimes |\epsilon_\mathbf{1}\rangle + |\epsilon_\mathbf{2}\rangle \otimes |\epsilon_\mathbf{2}\rangle \right] \ , \\
|\Psi_4\rangle &= \frac{1}{\sqrt{2}} \left[|\epsilon_\mathbf{1}\rangle \otimes |\epsilon_\mathbf{1}\rangle - |\epsilon_\mathbf{2}\rangle \otimes |\epsilon_\mathbf{2}\rangle \right] \ .
\end{aligned} \tag{3.6}$$

We choose $|\Psi_1\rangle$, $|\Psi_2\rangle$, $|\Psi_3\rangle$ and $|\Psi_4\rangle$ as an orthonormal basis (rather than $|\epsilon_{\mathbf{1}}\rangle\otimes|\epsilon_{\mathbf{1}}\rangle$, $|\epsilon_{\mathbf{1}}\rangle\otimes|\epsilon_{\mathbf{2}}\rangle$, $|\epsilon_{\mathbf{2}}\rangle \otimes |\epsilon_{\mathbf{1}}\rangle$ and $|\epsilon_{\mathbf{2}}\rangle \otimes |\epsilon_{\mathbf{2}}\rangle$) for two reasons. First, $|\Psi_1\rangle$ and $|\Psi_3\rangle$, but not $|\Psi_2\rangle$ or $|\Psi_4\rangle$, are invariant under rotations around the axis of symmetry – the axis along which the photons propagate. (See Prob. 3.7.) Therefore, the polarization state of the photons must be either $|\Psi_1\rangle$ or $|\Psi_3\rangle$. Second, the parity of the state $|\Psi_1\rangle$ is odd, while the parity of the states $|\Psi_2\rangle$, $|\Psi_3\rangle$ and $|\Psi_4\rangle$ is even. (See Prob. 3.8.) Pions and positronium have odd parity and decay by emitting photons in the polarization state $|\Psi_1\rangle$.

The scalar product of $|\Psi_1\rangle$ with $|\epsilon_{\mathbf{1}}\rangle \otimes |\epsilon_{\mathbf{1}}\rangle$ or $|\epsilon_{\mathbf{2}}\rangle \otimes |\epsilon_{\mathbf{2}}\rangle$ vanishes; hence in the state $|\Psi_1\rangle$, the photons have opposite polarizations. In the state $|\Psi_3\rangle$, which could arise from an atomic cascade, the photons have the *same* polarization, because the scalar product of $|\Psi_3\rangle$ with $|\epsilon_{\mathbf{1}}\rangle \otimes |\epsilon_{\mathbf{2}}\rangle$ or $|\epsilon_{\mathbf{2}}\rangle \otimes |\epsilon_{\mathbf{1}}\rangle$ vanishes. These correlations hold even if we rotate the polarization axes from $\epsilon_{\mathbf{1}}$, $\epsilon_{\mathbf{2}}$ to $\epsilon_{\mathbf{1}}{}'$, $\epsilon_{\mathbf{2}}{}'$,

$$\begin{aligned} |\epsilon_{\mathbf{1}}{}'\rangle &= |\epsilon_{\mathbf{1}}\rangle \cos\phi - |\epsilon_{\mathbf{2}}\rangle \sin\phi \ , \\ |\epsilon_{\mathbf{2}}{}'\rangle &= |\epsilon_{\mathbf{1}}\rangle \sin\phi + |\epsilon_{\mathbf{2}}\rangle \cos\phi \ , \end{aligned} \tag{3.7}$$

for $|\Psi_1\rangle$ looks just the same in the new basis:

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}} \left[|\epsilon_{\mathbf{1}}{}'\rangle \otimes |\epsilon_{\mathbf{2}}{}'\rangle - |\epsilon_{\mathbf{2}}{}'\rangle \otimes |\epsilon_{\mathbf{1}}{}'\rangle \right] \ . \tag{3.8}$$

So does $|\Psi_3\rangle$:

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}} \left[|\epsilon_{\mathbf{1}}{}'\rangle \otimes |\epsilon_{\mathbf{1}}{}'\rangle + |\epsilon_{\mathbf{2}}{}'\rangle \otimes |\epsilon_{\mathbf{2}}{}'\rangle \right] \ . \tag{3.9}$$

Equations (3.8–9) are just what rotational symmetry demands. An experiment in 1949 first checked and confirmed the anticorrelated polarizations of photons pairs from pion or positronium decay [7]. The experiment ruled out a model, suggested by Einstein, in which the state $|\Psi_1\rangle$ decays spontaneously to $|\epsilon_{\mathbf{1}}\rangle \otimes |\epsilon_{\mathbf{2}}\rangle$ or $|\epsilon_{\mathbf{2}}\rangle \otimes |\epsilon_{\mathbf{1}}\rangle$ (with equal probability) as the photons fly off [8]. Later experiments checked and confirmed the correlated polarizations of photon pairs emitted in an atomic cascade. How does the EPR claim apply to these experiments?

Let Alice and Bob help us with the measurements again. Pairs of photons fly off from an atomic cascade, with one photon in each pair arriving at Alice's polarizer and the other arriving at Bob's polarizer. After each arrival, Alice and Bob reset the pass axes of their polarizers. They may reset the pass axes at random, from a finite list of settings, or according to a common plan. Either way, they find that when the pass axes of their polarizers are parallel, both photons in each pair pass the polarizers, or both do not; when the pass axes are at right angles, only one photon in each pair passes a polarizer. Suppose that Bob sets the pass axis of his polarizer to $\epsilon_{\mathbf{1}}$. If Alice sets the pass axis of her polarizer to $\epsilon_{\mathbf{1}}$, and a photon passes through, the photon arriving at Bob's polarizer also passes through. If Alice does not set the pass axis of her polarizer to either $\epsilon_{\mathbf{1}}$ or $\epsilon_{\mathbf{2}}$, quantum theory does not predict whether the photon that arrives at Bob's polarizer passes through. What would Einstein, Podolsky and Rosen say? The result of Bob's measurement cannot depend on what Alice measures. Hence the photon arriving at Bob's polarizer must pass through anyway. Quantum theory is incomplete, because quantum theory fails to predict that the photon must pass through.

Indeed, EPR would claim that quantum theory is hopelessly incomplete. The conclusion that holds when the pass axis of Bob's polarizer is ϵ_1 holds for any setting of his polarizer. Similarly, it holds for any setting of Alice's polarizer. Hence a complete theory must predict the result of *any* polarization measurement by either Alice or Bob. But the photons cannot anticipate what Alice and Bob will measure, so the EPR claim implies that every pair of photons arrives at Alice's and Bob's respective laboratories with a complete – i.e. infinite – list of results, one for each measurement that Alice and Bob might make. Still, the EPR claim seems as untestable as Bishop Berkeley's claim that a tree does not fall if no one sees it fall.

In 1964, however, Bell published a remarkable paper that showed, in effect, how to test whether unseen trees fall [9]. He showed that the EPR claim implies an inequality that some quantum correlations do not satisfy. Bell's paper appeared in the first volume of a journal that folded almost immediately. Moreover his test, involving electron spin, was not practical at the time. But five years later, Clauser, Horne, Shimony, and Holt [10] (CHSH) generalized Bell's inequality; the CHSH inequality, applied to photon polarization, allows a practical test of the EPR claim. Let us derive the CHSH inequality.

We return to Alice and Bob and their measurements. According to EPR, every pair of photons represents a complete list of answers (results) for each pair of questions (observables) that Alice and Bob might ask (measure). We refer to such lists as local plans; let λ denote a local plan. Each local plan must list results for the observables that Alice and Bob may measure, and the set of observables is infinite. But let us consider just two possible observables, A and A', that Alice may measure, and two observables, B and B', that Bob may measure. Let $\rho(\lambda)$ denote the relative probability that a photon pair carries a local plan λ. We normalize $\rho(\lambda)$:

$$\int d\lambda \rho(\lambda) = 1 \ ,$$

where the integration is over all λ. Given a local plan λ, let $P(A; a; \lambda)$ be the probability that a measurement A yields the result a. Similarly, let $P(A, B; a, b; \lambda)$ be the probability that measurements A and B (on two photons) yield results a and b, respectively. The plan λ is local, hence $P(A, B; a, b; \lambda)$ factorizes:

$$P(A, B; a, b; \lambda) = P(A; a; \lambda) P(B; b; \lambda) \ . \tag{3.10}$$

Now let $P(A, B; a, b)$ be the probability that measurements of A and B on a photon pair yield a and b, respectively. It is the average of $P(A, B; a, b; \lambda)$ weighted by $\rho(\lambda)$, i.e.

$$P(A, B; a, b) = \int d\lambda \rho(\lambda) P(A, B; a, b; \lambda) \ .$$

We define the correlation between measurements A and B to be

$$C(A, B) = \sum_{ij} a_i b_j P(A, B; a_i, b_j) \ , \tag{3.11}$$

where the a_i and b_j are possible results of measurements A and B, respectively. If A and B are photon polarization or electron spin measurements, each has two possible results; but in

general, we let A and B have any finite number of possible results. Without loss of generality, we assume $-1 \le a_i, b_j \le 1$. We will prove that a combination S_{CHSH} of correlations,

$$S_{CHSH}(A, A'; B, B') = C(A, B) + C(A', B) + C(A, B') - C(A', B') \ ,$$

is bounded above and below:

$$-2 \le S_{CHSH}(A, A'; B, B') \le 2 \ . \tag{3.12}$$

To prove Eq. (3.12), we fix λ and look at the sum of products

$$\sum_{ij} a_i P(A; a_i; \lambda) \left[b_j P(B; b_j; \lambda) + b'_j P(B'; b'_j; \lambda) \right] + \sum_{ij} a'_i P(A'; a'_i; \lambda) \left[b_j P(B; b_j; \lambda) - b'_j P(B'; b'_j; \lambda) \right] \ . \tag{3.13}$$

The absolute values of $\sum_i a_i P(A; a_i; \lambda)$ and $\sum_j b_j P(B; b_j; \lambda)$ etc. are bounded by 1. Hence each line in Eq. (13.3) is bounded in absolute magnitude by 2. The sum of the two lines is also bounded by 2, because (for example) if $\sum_j b_j P(B; b_j; \lambda) + \sum_j b'_j P(B'; b'_j; \lambda)$ has magnitude 2, then $\sum_j b_j P(B; b_j; \lambda) - \sum_j b'_j P(B'; b'_j; \lambda)$ vanishes, and vice versa. (See Prob. 3.12.) Thus

$$-2 \le \sum_{ij} [a_i b_j P(A; a_i; \lambda) P(B; b_j; \lambda) + a'_i b_j P(A'; a'_i; \lambda) P(B; b_j; \lambda) + a_i b'_j P(A; a_i; \lambda) P(B'; b'_j; \lambda) - a'_i b'_j P(A'; a'_i; \lambda) P(B'; b'_j; \lambda)] \le 2 \ .$$

Multiplying by $\rho(\lambda)$ and integrating over λ, we obtain the CHSH inequality, Eq. (3.12):

$$-2 \le C(A, B) + C(A', B) + C(A, B') - C(A', B') \le 2 \ .$$

The CHSH inequality follows from the very assumption that local results exist, whether or not anyone measures them – that a tree falls whether or not anyone sees it fall. This reasonable ontological assumption underlies the EPR claim.

But some quantum correlations violate the CHSH inequality. To demonstrate a violation, we define the quantum correlation $C_Q(A, B)$ by replacing $P(A, B; a_i, b_j)$ in Eq. (3.11) with the quantum probability $P_Q(A, B; a_i, b_j)$:

$$C_Q(A, B) = \sum_{ij} a_i b_j P_Q(A, B; a_i, b_j) \ . \tag{3.14}$$

Alice and Bob measure linear polarizations in the plane of $\boldsymbol{\epsilon_1}$ and $\boldsymbol{\epsilon_2}$, with an angle θ_{AB} between the pass axes of A and B; let the values 1 and -1 correspond to a photon passing or not passing, respectively. We can obtain $C_Q(A, B)$ as follows. Suppose Bob's measurement yields 1, i.e. his photon passes his polarizer. The photons in each pair are correlated, hence Alice's photon is polarized parallel to Bob's. Then from Sect. 3.2 we infer that the probability that Alice's photon passes *her* polarizer is $\cos^2 \theta_{AB}$. By rotational symmetry, we have $P_Q(A, B, 1, 1) = (\cos^2 \theta_{AB})/2 = P_Q(A, B, -1, -1)$ and $P_Q(A, B, -1, 1) = (\sin^2 \theta_{AB})/2 = P_Q(A, B, 1, -1)$. Then $C_Q(A, B) = \cos^2 \theta_{AB} - \sin^2 \theta_{AB} = \cos 2\theta_{AB}$. In

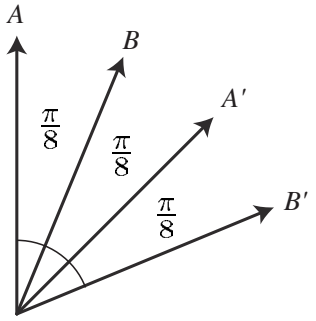

Figure 3.6: Polarizer pass axes in a test of the CHSH inequality, Eq. (3.12).

particular, let A, B, A' and B' correspond to polarizers with their pass axes in a common plane and an angle $\pi/8$ between the pass axes of A and B, B and A', and A' and B'. (See Fig. 3.6.) We have

$$C_Q(A,B) = C_Q(A,B') = C_Q(A',B) = \frac{\sqrt{2}}{2} = -C_Q(A',B')$$

and the sum

$$C_Q(A,B) + C_Q(A',B) + C_Q(A,B') - C_Q(A',B') = 2\sqrt{2}$$

violates the CHSH inequality, as we set out to demonstrate. Quantum correlations are nonlocal; they cannot arise from local plans.

An experimental test of the CHSH inequality, by Aspect, Dalibard, and Roger [11], measured correlations along the polarization axes of Fig. 3.6. The source of the photon pairs was an atomic cascade. In this experiment, the measurement settings switched pseudorandomly between A and A' and between B and B' in a time short compared to the time of flight of the photons. The correlations in this experiment were consistent with quantum correlations and violated the CHSH inequality by five standard deviations. In a more recent experiment by Wiehs et al. [12] the switching between these measurement settings was truly random and spacelike separated; the measured correlations violated the CSHS inequality by 30 standard deviations.

There are many extensions of Bell's inequality. Greenberg, Horne, and Zeilinger [13] (GHZ) found a remarkable extension involving *three* particles. (See Prob. 3.13.) The EPR paradox arises generically for any *entangled* state – any state of macroscopically separated systems that is not a product of states of each system. Any entangled state yields quantum correlations that violate a generalization of Bell's inequality [14].

3.5 Paradox and Beyond

The EPR claim assumes that Bob and Alice measure independent physical variables. Einstein, Podolsky and Rosen never anticipated that this reasonable assumption would prove inconsistent with experiment, that we cannot quite isolate systems in an entangled state from each other. But we cannot. As Bell put it, "The reasonable thing just doesn't work" [15].

Ironically, the claim that quantum theory is incomplete may well be correct, though not in the EPR sense. Quantum theory does not explain how we go from probability to observation, from possibility to actuality, as a complete theory would. There *is* such a complete theory, due to Bohm [16]. (See Prob. 9.3.) Bohm's theory is equivalent to quantum mechanics in its experimental predictions, but contains additional "hidden" variables that account for the results of every measurement. In keeping with Bell's inequality, these hidden variables are not local; so Einstein, Podolsky and Rosen would not have accepted them. But Bohm's theory is a complete quantum theory. Section 18.3 proposes final boundary conditions on the universe as a way to complete quantum theory. The hidden variables of this complete theory are nonlocal in *time*.

Problems

3.1 Consider the EPR paradox of Sect. 3.1. Does a measurement of x_1 affect a subsequent measurement of p_2? A measurement of x_1 affects the momentum component P of the measuring device according to the uncertainty relation $\Delta x_1 \Delta P \geq h$. Let m_2 and M denote the masses of particle 2 and of the measuring device, respectively. Let v_2 and $V = P/M$ denote the velocity components of particle 2 and of the measuring device, relative to a reference frame that does not accelerate during the measurement. Relative to the measuring device, the momentum of particle 2 is $p_2 = m_2(v_2 - V)$. Show that the uncertainty Δp_2 can be arbitrarily small, whatever Δx_1.

*3.2 (a) Show that a selector for polarization $|\mathbf{P}\rangle$ does not transmit any light of polarization $|\mathbf{P}'\rangle$ if $\langle \mathbf{P}|\mathbf{P}'\rangle = 0$. (See Sect. 3.2 and Fig. 3.5.)
(b) Show that Eq. (3.2) gives the fraction of light of polarization $|\mathbf{P}\rangle$ that passes a selector for polarization $|\mathbf{P}'\rangle$, and vice versa.

3.3 Let $A = |\psi_2\rangle\langle\psi_1|$ denote an operator such that $A|\phi\rangle = (\langle\psi_1|\phi\rangle)\,|\psi_2\rangle$. Show that A is a linear operator, i.e. that

$$A(c|\phi\rangle + c'|\phi'\rangle) = cA|\phi\rangle + c'A|\phi'\rangle \ .$$

Show that aA is a linear operator, if a is any complex number, and that any sum of linear operators is a linear operator.

3.4 The adjoint $A^\dagger$ of a linear operator A has the property that for any $|\phi\rangle$ and $|\phi'\rangle$,

$$\langle\phi|A^\dagger|\phi'\rangle = (\langle\phi'|A|\phi\rangle)^* \ .$$

(a) Show that if A is self-adjoint (or Hermitian), i.e. if $A = A^\dagger$, then the eigenvalues a_i of A are real.
(b) Show that if an operator U is unitary, i.e. $U^\dagger U = 1$, then its eigenvalues have absolute value 1.
(c) Unitary operators transform among incompatible bases. Show that if a set of vectors $|i\rangle$ is orthonormal, then so is the set of vectors $U|i\rangle$.

3.5 Let A be self-adjoint, with $A|\phi\rangle = a|\phi\rangle$ and $A|\phi'\rangle = a'|\phi\rangle$. Show that if $a \neq a'$, then $\langle\phi|\phi'\rangle = 0$.

3.6 An experiment to measure the spin component of an electron (along any axis) always yields $\hbar/2$ or $-\hbar/2$, corresponding to two basis vectors for the Hilbert space of electron spin. Let orthonormal vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ represent states with z-component of spin equal to $\hbar/2$ and $-\hbar/2$, respectively. Any other spin state is a superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$. Let $|u_1\rangle$ and $|u_2\rangle$ be two electron spin states with $\langle u_1|u_2\rangle = 0$; $|u_1\rangle$ and $|u_2\rangle$ are linear combinations of $|\uparrow\rangle$ and $|\downarrow\rangle$. Show that the entangled state

$$\frac{1}{\sqrt{2}}\left(|u_1\rangle \otimes |u_2\rangle - |u_2\rangle \otimes |u_1\rangle\right)$$

of two electrons equals the state

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\right) \tag{3.15}$$

up to an overall phase.

3.7 (a) Show that a ϕ rotation about the z-axis, Eq. (3.7), leaves $|\Psi_1\rangle$ and $|\Psi_3\rangle$ of Eqs. (3.5–6) invariant.
(b) Show that the quantum correlation for two photons in the state $|\Psi_1\rangle$ of Eq. (3.5) is $C_Q(A, B) = -\cos 2\theta_{AB}$, where θ_{AB} is the angle between the pass axes of linear polarizers in measurements A and B.

*3.8 (a) Show that the state $|\Psi_1\rangle$ in Eq. (3.5) is odd under parity and that the states $|\Psi_2\rangle$, $|\Psi_3\rangle$ and $|\Psi_4\rangle$ in Eq. (3.6) are even under parity. (Parity reverses the direction of the propagation vector $\mathbf{k}$ of each photon, as well as the polarization direction.)
(b) Show that Eq. (3.15) (in Prob. 3.6) is odd under parity. (Parity reverses the propagation direction of each electron, but not the spin.)

*3.9 A quantum wave function $\psi(\mathbf{x})$ is a state in an infinite-dimensional Hilbert space. We denote it $|\psi\rangle$. The scalar product of two quantum wave functions $\psi_1(\mathbf{x})$, $\psi_2(\mathbf{x})$ is $\langle\psi_1|\psi_2\rangle \equiv \int_{-\infty}^{\infty} \psi_1^*(\mathbf{x})\psi_2(\mathbf{x})d\mathbf{x}$. A basis for the Hilbert space may be discrete or continuous. The basis states $|\psi_i\rangle$ of a discrete basis satisfy $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, where $i, j = 1, 2, \ldots$. The basis states of a continuous basis satisfy an analogous condition, but with a δ-function in place of δ_{ij}. For example, $\delta(x - x_0)$ represents the state of a particle, in one space dimension, with position $x = x_0$. We denote it $|x_0\rangle$. We think of $\delta(x - x_0)$ as vanishing everywhere outside an arbitrarily small neighborhood of x_0. Formally, we define $\delta(x - x_0)$ via the integral

$$\int_a^b f(x)\delta(x - x_0)dx \ ,$$

which equals $f(x_0)$ if $a < x_0 < b$ and zero otherwise; $f(x)$ is an arbitrary differentiable function.

(a) A common representation of $\delta(x - x_0)$ is

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)} dk \ ;$$

that is,

$$\int_a^b f(x)\delta(x - x_0)dx = \frac{1}{2\pi} \lim_{M\to\infty} \int_a^b \int_{-M}^{M} f(x)e^{ik(x-x_0)} dkdx \ .$$

Show that the double integral yields $f(x_0)$ if $a < x_0 < b$.
(b) Show that $x|x_0\rangle = x_0|x_0\rangle$ and that $\langle x_i|x_j\rangle = \delta(x_i - x_j)$. In three space dimensions, $\delta(\mathbf{x} - \mathbf{x}_i) \equiv \delta(x - x_i)\delta(y - y_i)\delta(z - z_i)$ represents the state of a particle with position $\mathbf{x} = \mathbf{x}_i$, and we denote it $|\mathbf{x}_i\rangle$. Show that $\langle \mathbf{x}_i|\mathbf{x}_j\rangle = \delta(\mathbf{x}_i - \mathbf{x}_j)$. These conditions are continuous analogues of the discrete condition $\langle \psi_i|\psi_j\rangle = \delta_{ij}$. In keeping with Sect. 3.3, we have $|\langle \psi|\mathbf{x}_0\rangle|^2 = |\psi(\mathbf{x}_0)|^2$ as the probability density, in the state $|\psi\rangle$, to find the particle at a point $\mathbf{x}_0$. (It is the probability *density*, not the probability, because the state $|\mathbf{x}_0\rangle$ is not normalized, i.e. $\langle \mathbf{x}_0|\mathbf{x}_0\rangle \neq 1$.)
(c) The quantum wave $(2\pi\hbar)^{-1/2}e^{ip_0x/\hbar}$ represents the state of a particle, in one space dimension, with momentum $p = p_0$. (See Eq. (2.4).) We denote it $|p_0\rangle$. Show that $\langle p_i|p_j\rangle = \delta(p_i - p_j)$. (d) The analogue of Eq. (3.4) for the Hermitian operator p is

$$p = \int_{-\infty}^{\infty} |p_0\rangle\langle p_0|p_0 dp_0 \ .$$

Show that $p\psi(x) = -i\hbar d\psi(x)/dx$ where $\psi(x)$ is any function with a Fourier transform. Thus $-i\hbar d/dx$ represents the observable p.

3.10 For a given state $|\Psi\rangle$ and Hermitian operators A, B, let $\langle A\rangle$ denote the expectation value $\langle \Psi|A|\Psi\rangle$ and ΔA denote the uncertainty $\Delta A = \left(\langle A^2\rangle - \langle A\rangle^2\right)^{1/2}$.
(a) Prove that

$$A|\Psi\rangle = \langle A\rangle|\Psi\rangle + \Delta A|\Psi_\perp^A\rangle \ ,$$

where $|\Psi_\perp^A\rangle$ is some state orthogonal to $|\Psi\rangle$.
(b) Use this result to prove the general uncertainty relation, $\Delta A\Delta B \geq \frac{1}{2}\,|\langle [A, B]\rangle|$, where $[A, B] = AB - BA$ is the commutator of A and B.
(c) Derive $\Delta x\Delta p \geq \hbar/2$. (See Prob. 3.9(d).)

*3.11 Consider two macroscopically separated systems, S_A and S_B, prepared in an arbitrary state $|\Psi\rangle$. Let A be an observable on S_A and B be an observable on S_B. Show that the probability of any result of a measurement of A on S_A is independent of which observable B is measured on S_B. (Hint: Compute the probability of the joint result a_i for A and b_j for B and obtain the probability of the result a_i for A from it by summing over j.)

3.12 Rewrite Eq. (3.13) by defining

$$S^{(A)} = \sum_i a_i P(A; a_i; \lambda) \quad , \quad S^{(A')} = \sum_i a'_i P(A'; a'_i; \lambda) \ ,$$
$$S^{(B)} = \sum_j b_j P(B; b_j; \lambda) \quad , \quad S^{(B')} = \sum_j b'_j P(B'; b'_j; \lambda) \ ,$$

and

$$S = S^{(A)}[S^{(B)} + S^{(B')}] + S^{(A')}[S^{(B)} - S^{(B')}] \ .$$

Note that $-1 \leq S^{(A)}, S^{(A')}, S^{(B)}, S^{(B')} \leq 1$.
(a) Prove that if $S^{(A)}$, $S^{(A')}$, $S^{(B)}$ and $S^{(B')}$ all have absolute value 1, then $S^2 = 4$.
(b) Suppose that S is a maximum for $|S^{(A)}| < 1$. Prove $S^2 \leq 4$ using the stationary condition $\partial K / \partial S^{(A)} = 0$. Prove $|S| \leq 2$ in general.

*3.13 The GHZ state $\frac{1}{\sqrt{2}} (|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes |\uparrow\rangle_3 - |\downarrow\rangle_1 \otimes |\downarrow\rangle_2 \otimes |\downarrow\rangle_3)$ is a simultaneous eigenstate of $\sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)}$, $\sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)}$, $\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}$ and $\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}$, where the Pauli spin matrices $\sigma_x^{(j)}, \sigma_y^{(j)}$ act on the j-th state in each product and

$$\sigma_x^{(j)}|\uparrow\rangle_j = |\downarrow\rangle_j \ , \quad \sigma_x^{(j)}|\downarrow\rangle_j = |\uparrow\rangle_j \ ,$$
$$\sigma_y^{(j)}|\uparrow\rangle_j = i|\downarrow\rangle_j \ , \quad \sigma_y^{(j)}|\downarrow\rangle_j = -i|\uparrow\rangle_j \ .$$

What is the eigenvalue of each operator on the GHZ state? If the operators $\sigma_x^{(1)}$, $\sigma_x^{(2)}$, $\sigma_x^{(3)}$ and $\sigma_y^{(1)}$, $\sigma_y^{(2)}$, $\sigma_y^{(3)}$ all have simultaneous (hidden) values, what assignment of -1 or 1 to each is consistent with the quantum results?

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4 Phases and Gauges

Chapter 2 presents thought experiments in which electrons, after passing through a screen with two slits, display an interference pattern. Einstein claimed that, if we closely observe the recoil of the screen after each electron passes, we can determine through which slit the electron passed, and the interference pattern will remain. Bohr, however, insisted that quantum mechanics is consistent only if any procedure that reveals through which slit each electron passes also destroys the interference pattern. Bohr showed that Einstein's procedure for observing the recoil of the screen does, indeed, destroy the interference pattern. Since we claim that quantum mechanics is consistent, we must be able to show that any refinement or modification of Einstein's procedure would also destroy the interference pattern. We now describe two procedures, more sophisticated than Einstein's, to reveal through which of two slits an electron passes. Neither seems to destroy the interference pattern.

4.1 Two Paradoxical Procedures

In the first procedure, we put a capacitor next to the screen, halfway between the two slits. (See Fig. 4.1.) The flat plates of the capacitor are perpendicular to the screen and we assume that they are initially at rest. The plates, when charged, carry equal and opposite charges; each plate produces a constant electric field, perpendicular to the plates, but on either side of the capacitor these fields cancel. Thus, on either side of the capacitor, a passing electron feels no force. The electric field does fringe at the edges of the plates, but if we charge the capacitor only during the short time that the electron is on one side or the other of the capacitor, the electron never encounters this fringe field and never feels any force. However, the electron

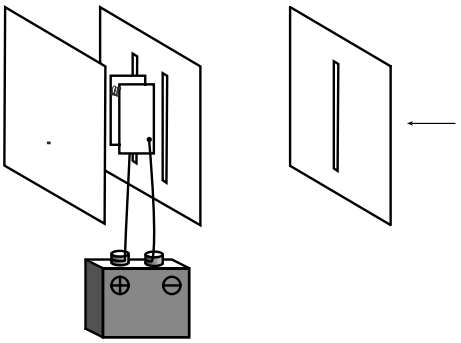

Figure 4.1: A two-slit experiment with a charged capacitor between the slits. An insulated spring keeps the plates apart.

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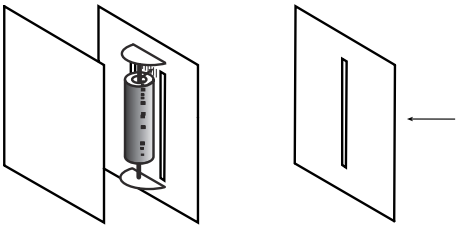


Figure 4.2: A two-slit experiment with a rotating, charged cylinder between the slits.

exerts a force on the plates of the capacitor: it repels the negatively charged plate, and attracts the positively charged plate. So it pulls the plates of the capacitor apart when it passes through one slit, and presses them together when it passes through the other slit. The behavior of the plates reveals through which slit the electron passed! But the plates exert no force either on the electron[1] or the screen (which is neutral). How can they affect the interference pattern?

In the second procedure a long cylinder, with a wire along its axis, takes the place of the flat plates. (See Fig. 4.2.) The wire is fixed in space and the cylinder is free to rotate about it. The wire and cylinder carry equal and opposite charges, distributed uniformly; outside the cylinder their electric fields cancel, and once again the electron feels no electric force. The cylinder, if it rotates, produces a magnetic field that vanishes outside it (like the magnetic field of a solenoid). So the electron also feels no magnetic force. However, the electron acts upon the cylinder. The passing electron generates a magnetic field, and the changing magnetic flux through the cylinder induces an electric field curling around it (by Faraday's law). This electric field changes the angular speed of the cylinder, but only briefly: the cylinder rotates freely with the same angular speed ω before and after the electron passes.

Can we detect the action of the electron? The cylinder has an arrow pointing out from its axis of rotation; it defines an angular position ϕ. Suppose we observe the arrow repeatedly *before* the electron passes, at times $t = 0, T, 2T, \ldots$, and find it pointing the same direction each time.[2] *After* the electron passes, we again observe the arrow at a time $t = nT$, where n is an integer, and find that the arrow has shifted direction slightly by an angle $\delta\phi$. A calculation (Prob. 4.1) shows that $\delta\phi$ is

$$\delta\phi = \pm\frac{\pi {r_c}^2 Qe}{c^2 Il} \ , \tag{4.1}$$

where r_c is the radius of the cylinder, Q is its charge, I is its moment of inertia and l its length. Equation (4.1) is valid when the minimum distance of the electron from the cylinder is small compared to l but large compared to r_c. The sign of $\delta\phi$ depends on whether the electron passes to the right or to the left of the cylinder. By Lenz's law, the angular speed of the cylinder

[1] The plates of a capacitor conduct, so an image charge in the nearest plate attracts the electron. The attraction is of order e^2 (the electron and its image have charge $\pm e$) hence negligible compared to the force on the plates. We can remove this attraction by replacing the capacitor with oppositely charged, nonconducting plates; then there is no image charge. We separate the plates only when the electron is on one side or the other of the plates, so the electron also never encounters a fringe field.

[2] We do not thereby measure the angular speed ω, for we do not know how many times the cylinder rotates during a time T; we only know that ω must be a multiple of $2\pi/T$. In quantum theory, the angular momentum of a freely rotating cylinder must be a multiple of $\hbar$; hence ω must be a multiple of $\hbar/I$, where I is the moment of inertia of the cylinder.

changes so as to cancel the magnetic flux due to the electron. The behavior of the cylinder reveals through which slit the electron passed! Yet the cylinder hardly exerts a force on the electron or on the screen (if I is large): the electric field due to the changing angular speed of the cylinder is proportional to $1/I$. How can the cylinder affect the interference pattern?

These two thought experiments challenge our claim that quantum mechanics is consistent. If we cannot show that the procedures destroy the electron's interference pattern, we must conclude that quantum mechanics is inconsistent. We have noted the fringing of electric and magnetic fields. However, in the first procedure we could charge the capacitor plates for a short time only, while the electron passes near the center of the plates and is far from the edges. In the second procedure, we can make the cylinder and wire as long as we like. Thus, we really can neglect fringing. The resolution of this paradox is more subtle and remarkable, requiring insight into the role of electromagnetic potentials in quantum mechanics.

4.2 Classical and Quantum Phases

Interference is familiar from the classical physics of waves, but there is a fundamental difference between classical and quantum waves. Typical classical waves are water waves, sound waves and light waves. Each is defined by variation in space and time of a measurable quantity: the water level, the density of a gas, the strengths of electric and magnetic fields. These quantities define not only the length and frequency of a wave, but also its phase. By contrast, *nothing* that we can measure defines the phase of a quantum wave. We can measure only phase *differences*. When a light wave passes through a screen with two slits, we can determine the phase of each partial wave emerging from the slits. When an electron passes through a screen with two slits, we can only determine the relative phase of the two parts of the wave function emerging from the slits. Mathematically, the quantum wave $\Psi(\mathbf{x}, t)$ consists of a modulus $n(\mathbf{x}, t)$ and a phase $\varphi(\mathbf{x}, t)$, both real functions of $\mathbf{x}$ and t:

$$\Psi(\mathbf{x}, t) = n(\mathbf{x}, t)e^{i\varphi(\mathbf{x},t)} \ .$$

The probability density $\rho(\mathbf{x}, t)$ to find the particle at a point $\mathbf{x}$ at time t is

$$\rho(\mathbf{x}, t) = |\Psi(\mathbf{x}, t)|^2 = [n(\mathbf{x}, t)]^2 \ .$$

(See Prob. 3.9(b).) In interference phenomena, two partial quantum waves sum,

$$\Psi(\mathbf{x}, t) = \Psi_1(\mathbf{x}, t) + \Psi_2(\mathbf{x}, t) = n_1(\mathbf{x}, t)e^{i\varphi_1(\mathbf{x},t)} + n_2(\mathbf{x}, t)e^{i\varphi_2(\mathbf{x},t)} \ ,$$

and the probability density depends on the relative phase $\varphi_1(\mathbf{x}, t) - \varphi_2(\mathbf{x}, t)$:

$$\begin{aligned}\rho(\mathbf{x}, t) &= |n_1(\mathbf{x}, t)e^{i\varphi_1(\mathbf{x},t)} + n_2(\mathbf{x}, t)e^{i\varphi_2(\mathbf{x},t)}|^2 \\ &= n_1^2 + n_2^2 + 2n_1n_2\cos\left[\varphi_1(\mathbf{x}, t) - \varphi_2(\mathbf{x}, t)\right] \ .\end{aligned}$$

Thus the phase of $\Psi_2(\mathbf{x}, t)$ relative to the phase of $\Psi_1(\mathbf{x}, t)$ appears directly in the electron's interference pattern; it is not arbitrary. On the other hand, the absolute phase of the quantum wave is arbitrary, in the following sense: if we multiply $\Psi(\mathbf{x}, t)$ by an overall complex phase factor $e^{i\lambda}$ (with λ real and constant), the probability density does not change:

$$\rho(\mathbf{x}, t) = |e^{i\lambda}\Psi(\mathbf{x}, t)|^2 = |\Psi(\mathbf{x}, t)|^2 \ .$$

Indeed, in this sense, the absolute phase of a quantum wave at *each* spacetime point is arbitrary: if we multiply $\Psi(\mathbf{x},t)$ by an overall complex phase factor $e^{i\lambda(\mathbf{x},t)}$ (with λ a real function of $\mathbf{x}$ and t) the probability density does not change:

$$\rho(\mathbf{x},t) = |e^{i\lambda(\mathbf{x},t)}\Psi(\mathbf{x},t)|^2 = |\Psi(\mathbf{x},t)|^2 \ .$$

We call the transformation

$$\Psi(\mathbf{x},t) \to \Psi'(\mathbf{x},t) = e^{i\lambda}\Psi(\mathbf{x},t) \tag{4.2}$$

with λ constant a *global* phase transformation and the transformation

$$\Psi(\mathbf{x},t) \to \Psi'(\mathbf{x},t) = e^{i\lambda(\mathbf{x},t)}\Psi(\mathbf{x},t) \tag{4.3}$$

a *local* phase transformation. The probability density is invariant under both. The interference pattern is invariant under both because, at a given spacetime point $(\mathbf{x},t)$, the same factor $e^{i\lambda(\mathbf{x},t)}$ multiplies $\Psi_1(\mathbf{x},t)$ and $\Psi_2(\mathbf{x},t)$. But some expectation values are not invariant under local phase transformations. For example, the components of momentum $\mathbf{p}$ are represented by linear (differential) operators $-i\hbar\partial/\partial x$, $-i\hbar\partial/\partial y$ and $-i\hbar\partial/\partial z$. (See Prob. 3.9(d).) The expectation value $\langle\Psi|\mathbf{p}|\Psi\rangle$ is *not* invariant under local phase transformations. On the other hand, all expectation values are invariant under global phase transformations.

4.3 Phase Meets Gauge

The apparent conclusion of the previous section is that a global phase transformation, applied to a quantum wave, yields an equivalent wave; but a local phase transformation, applied to a quantum wave, yields (in general) an inequivalent wave. At any one point in spacetime we can choose the phase of the wave function, but that choice fixes the phase of the wave function at every other point. Nevertheless, this conclusion is not valid if we include electromagnetic interactions in quantum mechanics. We will now see that, at least for charged particles, the quantum phase is arbitrary at *every* point in spacetime, so two quantum waves related by a local phase transformation are indeed equivalent.

In classical physics, the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ obey four equations – Maxwell's equations – which describe how each field depends on the other and on charges (including moving charges). A fifth equation – the Lorentz force equation – describes how a charged particle reacts to electric and magnetic fields. All the quantities appearing in these five equations are measurable, and there are no other measurable electromagnetic quantities. Even so, two auxiliary quantities are in common use: a scalar field $V(\mathbf{x},t)$ and a vector field $\mathbf{A}(\mathbf{x},t)$, which together form a four-vector. V and $\mathbf{A}$ completely determine $\mathbf{E}$ and $\mathbf{B}$ according to the equations[3]

$$\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} - \nabla V \ , \quad \mathbf{B} = \nabla \times \mathbf{A} \ . \tag{4.4}$$

[3] We use Gaussian units.

$\mathbf{E}$ and $\mathbf{B}$, however, do *not* determine V and $\mathbf{A}$. Suppose we modify V and $\mathbf{A}$ as follows:

$$V \to V - \frac{1}{c}\frac{\partial}{\partial t}\Lambda \ , \quad \mathbf{A} \to \mathbf{A} + \nabla\Lambda \ , \tag{4.5}$$

where $\Lambda(\mathbf{x}, t)$ is any scalar function of space and time. This modification of V and $\mathbf{A}$ does nothing to $\mathbf{E}$ and $\mathbf{B}$, according to Eq. (4.4). Since only $\mathbf{E}$ and $\mathbf{B}$ are directly measurable, the transformation Eq. (4.5) has no observable consequences, and V and $\mathbf{A}$ are to some extent arbitrary. They are arbitrary in the same way that the quantum phase $\varphi(\mathbf{x}, t)$ is arbitrary: at any spacetime point $(\mathbf{x}, t)$, we cannot measure their values. We can only measure (to an extent) their spacetime dependence. For example, only the difference in V from one point in space to another has observable consequences; V alone does not. Thus, the transformation Eq. (4.5) is analogous to a change in the initial reading on a gauge (where the difference between final and initial readings yields a measurement) and is called a *gauge* transformation.

If V and $\mathbf{A}$ are somewhat arbitrary, why bother with them? One reason is that it may be simpler to solve Maxwell's equations for the four auxiliary quantities V and $\mathbf{A}$ than for the six physical quantities $\mathbf{E}$ and $\mathbf{B}$. An additional reason is that we may want to obtain the Lorentz force equation (for a particle of mass m and charge e)

$$m\frac{d^2\mathbf{x}}{dt^2} = e\mathbf{E} + \frac{e}{c}\left(\frac{d\mathbf{x}}{dt}\right) \times \mathbf{B} \tag{4.6}$$

from a Hamiltonian. $\mathbf{E}$ and $\mathbf{B}$ cannot appear in the Hamiltonian, for a Hamiltonian containing $\mathbf{E}$ and $\mathbf{B}$ would yield an equation of motion containing their derivatives. However, Hamilton's equations for a Hamiltonian in which V and $\mathbf{A}$ appear,

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + eV \ , \tag{4.7}$$

yield the Lorentz force equation, Eq. (4.6). The strongest reason for introducing V and $\mathbf{A}$ is the Hamiltonian itself. The Hamiltonian – the sum of the kinetic and potential energies – is a conserved quantity. Without the electromagnetic potentials we could not arrive at a notion of conservation. Still, in classical mechanics we can choose either Eq. (4.7) or Eq. (4.6) as a starting point.

In quantum mechanics we cannot choose; we *must* start with a Hamiltonian.[4] Any quantum state $|\Psi\rangle$ of a system satisfies the Schrödinger equation $i\hbar\partial|\Psi\rangle/\partial t = H|\Psi\rangle$, where H is the Hamiltonian operator for the system. The Schrödinger equation corresponding to Eq. (4.7) is

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x}, t) = \left[\frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 + eV\right]\Psi(\mathbf{x}, t) \tag{4.8}$$

and describes a charged, nonrelativistic quantum particle in a vector potential $\mathbf{A}$ and a scalar potential V. Applying the gauge transformation Eq. (4.5), we arrive at a transformed Hamiltonian

[4]The Hamiltonian is the starting point for both Schrödinger's equation and Heisenberg's matrix mechanics. Feynman's path integral is based on the Lagrangian (in which V and $\mathbf{A}$ also appear); but in general the Hamiltonian is essential for obtaining the correct path integral. (See Sect. 12.3.) Regarding a gauge-invariant form of the Schrödinger equation, see Prob. 4.2.

which must describe the same particle in the same magnetic and electric fields:

$$i\hbar\frac{\partial}{\partial t}\Psi'(\mathbf{x},t) = \left[\frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A} - \frac{e}{c}\nabla\Lambda\right)^2 + eV - \frac{e}{c}\frac{\partial}{\partial t}\Lambda\right]\Psi'(\mathbf{x},t)\ . \tag{4.9}$$

We have written $\Psi(\mathbf{x},t)$ as the solution of Eq. (4.8) and $\Psi'(\mathbf{x},t)$ as the solution of Eq. (4.9) because they are not the same wave function. Indeed,

$$\Psi'(\mathbf{x},t) = e^{ie\Lambda/\hbar c}\Psi(\mathbf{x},t)\ ; \tag{4.10}$$

we see by substitution that if $\Psi(\mathbf{x},t)$ is a solution of Eq. (4.8), then $\Psi'(\mathbf{x},t)$ is a solution of Eq. (4.9). Equation (4.10) is a local phase transformation as defined in Eq. (4.3), with $\lambda(\mathbf{x},t) = e\Lambda(\mathbf{x},t)/\hbar c$. Since $\Psi(\mathbf{x},t)$ and $\Psi'(\mathbf{x},t)$ have the same physical significance, it is apparent that the local phase transformation Eq. (4.10) *does* lead to an equivalent wave function, and that the phase of a quantum wave at *any* point is arbitrary, since $\Lambda(\mathbf{x},t)$ is arbitrary.

Thus, what we have called local phase transformations and gauge transformations (also called gauge transformations of the first and second kind, respectively) go hand in hand. As noted above, gauge transformations do not affect either probability densities or interference. What about expectation values? We find that all measurable quantities are gauge invariant; gauge transformations do not affect their expectation values. For example, the velocity of a particle with Eq. (4.7) as its Hamiltonian is

$$\frac{d}{dt}\langle\Psi|\mathbf{x}|\Psi\rangle = \frac{i}{\hbar}\langle\Psi|[H,\mathbf{x}]|\Psi\rangle = \frac{1}{m}\langle\Psi|\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)|\Psi\rangle\ ,$$

and is gauge invariant. (See Prob. 4.4.)

4.4 The Aharonov–Bohm Effect

Quantum mechanics needs V and $\mathbf{A}$ in a way that classical physics does not. The scalar and vector potentials appear in Schrödinger's equation, in which gauge transformations (of the potentials) combine with local phase transformations (of the quantum wave). The fact that these transformations go hand in hand gives the electromagnetic potentials greater significance in quantum physics than in classical physics. We began this chapter with two thought experiments involving electric and magnetic fields. We now return to the two experiments and consider the role of the electromagnetic potentials. We will find that V and $\mathbf{A}$ seem to influence the electron even where the electric and magnetic fields vanish! The first to realize the importance of this purely quantum effect of electromagnetic potentials, in 1959, were Aharonov and Bohm [1].

In the first experiment, a capacitor lies midway between two passing wave packets of a single electron.[5] Briefly, as they pass, we apply a potential difference to the capacitor. Let the potential vanish on the left plate of the capacitor and equal $V(t)$ on the right plate, and let $V(t)$ be zero except when the wave packets pass near the center of the plates. Thus neither wave packet ever encounters an electric field. (We could just as well let the potential equal $-V(t)$ on the left plate and vanish on the right plate; all that matters is the potential difference across the

[5] Section 5.2 presents examples of wave packets.

plates, and we can never make the potential vanish on both plates.) The wave packet $\Psi_L^0(\mathbf{x},t)$ passing to the left of the capacitor satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi_L^0(\mathbf{x},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi_L^0(\mathbf{x},t)\ , \tag{4.11}$$

while the wave packet $\Psi_R(\mathbf{x},t)$ passing to the right of the capacitor satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi_R(\mathbf{x},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi_R(\mathbf{x},t) + eV(t)\Psi_R(\mathbf{x},t)\ . \tag{4.12}$$

Although there is no electric field on either side of the capacitor, we cannot eliminate the scalar potential from *both* Eq. (4.11) and Eq. (4.12) because it does not vanish everywhere. We can, however, write $V(t)$ in Eq. (4.12) as a gauge transform of the zero potential by defining

$$\Lambda = -c\int^t V(t')dt'\ ,$$

so that $V(t) = -(1/c)\partial\Lambda/\partial t$. Now suppose Ψ_R^0 represents the solution of Eq. (4.12) with $V(t) = 0$. Then from Eq. (4.10) we obtain immediately

$$\Psi_R(\mathbf{x},t) = \Psi_R^0(\mathbf{x},t)e^{ie\Lambda/\hbar c} = \Psi_R^0(\mathbf{x},t)e^{-i(e/\hbar)\int^t V(t')dt'}\ .$$

Thus the *potential* difference between the plates induces a *phase* difference between the two wave packets. If the two wave packets arrive at the capacitor in a superposition $(\Psi_L^0+\Psi_R^0)/\sqrt{2}$, they leave the capacitor in the superposition

$$\frac{1}{\sqrt{2}}\left(\Psi_L^0 + e^{ie\Lambda/\hbar c}\Psi_R^0\right)\ ,$$

and the phase difference between the two wave packets will be observable as a *shift* in the lines of the interference pattern from their position when there is no potential difference. Note, however, that a relative phase of 2π is unobservable because one line merely takes the place of its neighbor in the interference pattern. (See Prob. 4.7.)

Thus the capacitor shifts the electron's interference pattern! This remarkable effect depends on the topology of space. The space available to the electron is not simply connected. The electron's paths from the two slits lie on one side or the other of a region where the potential is not constant in space. Yet along each path, the potential is constant.

In the second experiment, too, the region where the electron can go, where both electric and magnetic fields vanish, is not simply connected. The electron is forbidden to go into the cylinder. Although we can choose the scalar potential to vanish outside the cylinder, we cannot, in general, choose the vector potential $\mathbf{A}$ to vanish everywhere outside the cylinder. If it did, the integral of the vector potential along any closed path around the cylinder,

$$\Phi_B = \oint \mathbf{A}\cdot d\mathbf{x}\ ,$$

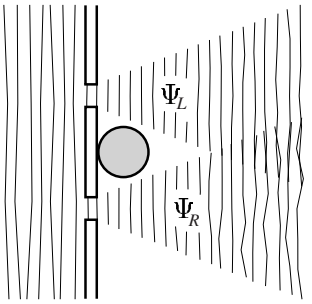


Figure 4.3: Wave packets overlapping in a space that is not simply connected.

would vanish. But the integral equals the enclosed magnetic flux Φ_B, which is nonzero if the cylinder rotates.

The topology common to the two experiments leads us to consider, as before, the Schrödinger equation satisfied by two wave packets, $\Psi_L(\mathbf{x},t)$ and $\Psi_R(\mathbf{x},t)$, passing through the left and right slits, respectively. $\Psi_L(\mathbf{x},t)$ passes the cylinder on the left and satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi_L(\mathbf{x},t) = \frac{1}{2m}\Big(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\Big)^2\Psi_L(\mathbf{x},t) \tag{4.13}$$

in the region to the left of the cylinder, while $\Psi_R(\mathbf{x},t)$ passes the cylinder on the right and satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi_R(\mathbf{x},t) = \frac{1}{2m}\Big(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\Big)^2\Psi_R(\mathbf{x},t) \tag{4.14}$$

in the region to the right of the cylinder. The left and right regions overlap where the wave packets separate and where they recombine. (See Fig. 4.3.) To specify $\mathbf{A}$, it is convenient to use cylindrical coordinates (r,θ,z) with the axis of the charged cylinder defining the z-axis.[6] A possible choice for $\mathbf{A}$ in the region outside the cylinder is $\mathbf{A} = \Phi_B\hat{\theta}/2\pi r$, where $\hat{\theta}$ is a unit vector is the θ-direction.

For this choice of vector potential, we can write $\mathbf{A}(\mathbf{x},t) = (\Phi_B/2\pi)\nabla\theta$, or $\mathbf{A} = \nabla\Lambda$ with $\Lambda = \Phi_B\theta/2\pi$. The coordinate θ increases by 2π with every full turn about the z-axis, and so Λ must be multivalued in the region outside the cylinder. In this region, then, $\mathbf{A} = \nabla\Lambda$ is not a gauge transform of zero vector potential, because Λ is not a single-valued function of space and time everywhere outside the cylinder. But the region to the left of the cylinder is simply connected, and also the region to the right. Within each region, Λ is a single-valued function of space and time, and $\mathbf{A} = \nabla\Lambda$ is a gauge transform of zero. Thus if Ψ_L^0 represents the solution to Eq. (4.13) when $\mathbf{A} = 0$, we know from Eqs. (4.8–10) that the solution to Eq. (4.13) for nonzero $\mathbf{A}$ is

$$\Psi_L = \Psi_L^0 e^{ie\Lambda/\hbar c} = \Psi_L^0 e^{ie\Phi_B\theta/2\pi\hbar c}\ .$$

Similarly, if Ψ_R^0 represents the solution to Eq. (4.14) when $\mathbf{A} = 0$, then the solution to Eq. (4.14) for nonzero $\mathbf{A}$ is

$$\Psi_R = \Psi_R^0 e^{ie\Lambda/\hbar c} = \Psi_R^0 e^{ie\Phi_B\theta/2\pi\hbar c}\ .$$

[6] Here for cylindrical coordinates we define $r = (x^2+y^2)^{1/2}$ and $\theta = \arctan(y/x)$ whereas for spherical coordinates we define $r = (x^2+y^2+z^2)^{1/2}$, $\theta = \arccos(z/r)$ and $\phi = \arctan(y/x)$.

Suppose that, without a vector potential, the partial waves from the two slits would recombine in the superposition

$$\frac{1}{\sqrt{2}}\left(\Psi_L^0 + \Psi_R^0\right) \ .$$

Because of the vector potential, they recombine in the superposition

$$\frac{1}{\sqrt{2}}\left(\Psi_L^0 + e^{ie\Phi_B/\hbar c}\Psi_R^0\right)$$

(up to an overall phase), since the difference in θ between the two partial waves is 2π. The vector potential induces a relative phase of magnitude $e\Phi_B/\hbar c$ between the partial waves. This relative phase shifts the interference pattern by an amount proportional to the flux Φ_B. Again, only the modular part of the phase, $e\Phi_B/\hbar c \bmod 2\pi$, is observable. (See Prob. 4.7.) A magnetic flux influences the electron's interference pattern even though the electron does not pass through any magnetic field! Experiments have confirmed this shift in the interference pattern [2].

4.5 Quantum Consistency and the Aharonov–Bohm Effect

The Aharonov–Bohm effect is the key to resolving the paradoxes presented at the beginning of this chapter. It shows us that a capacitor or a solenoid placed between two slits can affect the interference pattern of an electron diffracting through the two slits. We will now see how this effect preserves the consistency of quantum mechanics.

In the first paradoxical procedure described in Sect. 4.1, the electron's path influences the relative momentum of the two capacitor plates. To detect through which slit the electron passes, we must be able to measure the relative momentum of the plates to good accuracy. But the scalar potential between the capacitor plates is proportional to their separation. The more accurately we measure the relative momentum of the plates, the less accurately we can measure their relative position, and the more uncertain is the scalar potential between the plates. The Aharonov–Bohm effect implies that an uncertain scalar potential induces an uncertain relative phase between two paths passing on opposite sides of the capacitor, washing out the interference pattern.

This resolution of the paradoxes is qualitatively correct. Is it quantitatively correct, too? Let us consider how accurately we need to measure the relative momentum of the plates. If p_L and p_R represent the momentum of the left and right plates, respectively, we must determine whether the passing electron increases or decreases their relative momentum $p_R - p_L$. (The charged plates in any case attract one another, but let us assume that we can isolate the change in $p_R - p_L$ due to the electron.) The absolute value of the change in $p_R - p_L$, then, must be large compared to $\Delta(p_R - p_L)$, the accuracy with which we can measure $p_R - p_L$. It must be large if we are to be fairly certain about which way the electron went. The change in $p_R - p_L$ is the time integral of the force, $eE(t)$, so

$$|e\int dtE(t)| \gg \Delta(p_R - p_L) \ .$$

The relative phase between paths of the electron is $(e/\hbar)\int dtV(t)$, where $V(t)$ is the potential difference between the two plates at time t; $V(t)$ is proportional to the separation $x_R - x_L$ of the plates:

$$V(t) = E(t)(x_R - x_L) \ .$$

The uncertainty in $V(t)$ is at least $|E(t)|\Delta(x_R - x_L)$, so the phase uncertainty is at least

$$\Delta(\text{phase}) = (e/\hbar)|\int dtE(t)|\Delta(x_R - x_L) \gg \Delta(p_R - p_L)\Delta(x_R - x_L)/\hbar \ .$$

We now apply a form of Heisenberg's uncertainty relation. (See Sect. 2.3.) Prob. 3.10 presents the following theorem:

$$\Delta A \Delta B \geq \frac{1}{2}\,|\langle [A, B]\rangle| \ , \tag{4.15}$$

where A and B are any two Hermitian operators. Since $[x_R - x_L, p_R - p_L] = [x_R, p_R] + [x_L, p_L] = 2i\hbar$, we have $\Delta(\text{phase}) \gg \Delta(\mathrm{p_R} - \mathrm{p_L})\Delta(\mathrm{x_R} - \mathrm{x_L})/\hbar \geq 1$. Thus the uncertainty in the phase is large compared to 1 and washes out the interference pattern.

The Aharonov–Bohm effect applies in a similar way to the second paradoxical procedure. To detect which way the electron went, we must measure $\delta\phi$, the change in the angular position of the cylinder. The sign of $\delta\phi$ tells us which way the electron went; hence the uncertainty in ϕ must not be greater than $\delta\phi$. Equation (4.1) then implies

$$\Delta\phi \ll \frac{\pi {r_c}^2 Qe}{c^2 Il} \ . \tag{4.16}$$

On the other hand, the rotating cylinder produces a flux Φ_B

$$\Phi_B = 2\pi {r_c}^2 QL_z/Ilc \ , \tag{4.17}$$

where L_z is the angular momentum of the cylinder. In accord with the Aharonov–Bohm effect, it yields an electron phase of $(e/\hbar c)(2\pi {r_c}^2 QL_z/Ilc)$. The uncertainty in this phase is at least

$$\Delta(\text{phase}) \geq 2\pi {r_c}^2 eQ(\Delta L_z)/\hbar Ilc^2 \ . \tag{4.18}$$

The commutator of ϕ and L_z is $[\phi, L_z] = i\hbar$. (See Prob. 4.8.) Then Prob. 3.10 implies that $\Delta L_z \Delta\phi \geq \hbar/2$, and from Eqs. (4.16) and (4.18) we again obtain $\Delta(\text{phase}) \gg 1$. So here, too, detecting though which slit the electron went destroys the electron interference.[7]

We see that the Aharonov–Bohm effect is crucial to the consistency of quantum mechanics. Indeed, after Aharonov and Bohm predicted the effect, and while their prediction was still controversial, Furry and Ramsey [3] formulated paradoxes similar to those of Sect. 4.1 and applied the Aharonov–Bohm effect to resolve them, to show that the effect was not only correct, but necessary for consistency.

[7] An application of modular variables shows that the phase uncertainty in both procedures is actually 2π. See Prob. 5.3.

We close this section with a comment on the interpretation of the AB effect. Electromagnetic potentials have a significance in quantum theory that they lack in classical theory. We might conclude that in classical theory, $\mathbf{E}$ and $\mathbf{B}$ are the only physical variables, whereas in quantum theory, $\mathbf{A}$ and V are physical variables as well. But in quantum theory, as in classical theory, only gauge-invariant quantities (such as the total flux enclosed by two paths, or a difference in scalar potential) have physical meaning. Measurable quantities are gauge-invariant. We cannot measure V and $\mathbf{A}$ themselves. To define them to be physical variables contradicts the demand (in Chap. 7) for the closest possible correspondence between what theory describes and what we can measure. (Of course, V and $\mathbf{A}$ are physical in that they appear in physical equations, but they are not measurable.) Thus, instead of concluding that $\mathbf{A}$ and V are physical variables in quantum mechanics, we state a conclusion that later chapters develop: Only $\mathbf{E}$ and $\mathbf{B}$ are physical quantities, but they act nonlocally – a magnetic field *here* has physical effects on electrons *there*, and so on. Such action at a distance by a field is completely nonclassical.

4.6 Flux Quantization

As a simple application of the Aharonov–Bohm effect, let us consider the energy of a particle of charge q and mass M moving on a unit circle, when the circle encloses a solenoid. The particle has a single degree of freedom, its angular position on the circle, which we denote θ. The solenoid contains a magnetic flux Φ_B, but the particle never encounters a magnetic field; the field vanishes on the circle. We obtain the energies E_n and corresponding basis states $|\psi_n\rangle$ of the particle from the eigenvalue equation

$$H|\psi_n\rangle = E_n|\psi_n\rangle \ ,$$

where the Hamiltonian, Eq. (4.7), reduces to

$$H = \frac{1}{2M}\left(p_\theta - \frac{q}{c}A_\theta\right)^2 = \frac{1}{2M}\left(-i\hbar\frac{\partial}{\partial\theta} - \frac{q}{c}\frac{\Phi_B}{2\pi}\right)^2$$

in the gauge and cylindrical coordinates of Sect. 4.4. The solutions are

$$\psi_n(\theta) = \frac{1}{\sqrt{2\pi}}e^{in\theta}$$

where n is an integer; the eigenstates $|\psi_n\rangle$ form an orthonormal basis. The corresponding eigenvalues are $E_n = (n\hbar - q\Phi_B/2\pi c)^2/2M$. Any other state of the particle is a linear combination of these basis states.[8]

What is striking is that the energies $E_n = (n\hbar - q\Phi_B/2\pi c)^2/2M$ depend on the magnetic flux through the circle, even though the particle on the circle never encounters the flux. The ground-state (lowest) energy of the particle is $(n_0\hbar - q\Phi_B/2\pi c)^2/2M$, where n_0 is the integer nearest to $q\Phi_B/hc$; it is positive unless Φ_B is a multiple of hc/q.

A more realistic Hamiltonian for a charged particle contains additional degrees of freedom, and potential as well as kinetic energy operators; hence the eigenstates are more complicated,

[8]The states $|\psi_n\rangle$ are also eigenstates of angular momentum p_θ with eigenvalues $n\hbar$.

and the corresponding eigenvalues contain additional kinetic and potential energy terms. But the eigenvalues still depend periodically on the flux, and this dependence has experimental consequences. One consequence is the quantization of flux in a superconducting ring [4]. Experiments show that the flux through a cylindrical superconductor is quantized in multiples of $\Phi_B = hc/2e$. At first, these experiments seem to contradict our assumption that we can impose an arbitrary flux Φ_B in the region inaccessible to the charged particle. But what happens is that the imposed flux induces surface currents in the superconductor, circulating around the inaccessible region. These currents round off the imposed flux to a multiple of $hc/2e$ and thereby lower the average ground-state energy of all the particles. (The flux is quantized in multiples of $hc/2e$ because in a superconductor the effective particles are pairs of electrons, with charge $q = 2e$.)

4.7 Magnetoresistance

Figure 4.4 shows two wires, of resistances R_+ and R_-, connected in parallel. The familiar formula $1/R = 1/R_+ + 1/R_-$ implies that the resistance R between the two leads in Fig. 4.4 depends only on R_+ and R_-. Yet the topology of Fig. 4.4 is the topology of the two-slit experiment of Fig. 2.3: two electron paths make a closed loop. Could electrons in the two wires interfere, like electrons in the two slits? Perhaps the resistance between the leads depends on interference – destructive interference increases, and constructive interference decreases, the resistance.

To answer this question, we must consider more than topology. Electrons pass through the thought experiment of Fig. 2.3 one at a time, while the electrons in Fig. 4.4 scatter off each other and off atoms in the conductor. Inelastic scattering changes the energy, and thus the wavelength, of electrons in a conductor; then the spread in the wavelengths of the electrons washes out interference. Elastic scattering, however, does not change the energy of the electrons. It may change the phase of the electrons, but the change is the same for all electrons in the same state. (See Prob. 6.1 for an example of elastic scattering off a wall.) If scattering of at least some electrons in the wires is elastic, the interference should not all wash out.

The pattern of constructive and destructive interference in the two-slit experiment is predictable. It is not so easy to predict how the resistance of the loop in Fig. 4.4 should depend on the interference of the electrons in the two wires; we can, however, predict that the resistance must be periodic in the relative phase of the electrons. In particular, if the resistance depends on the magnetic flux through the loop, the dependence must be periodic with period hc/e, just as the dependence of the Aharonov–Bohm phase on flux is periodic with period hc/e. This

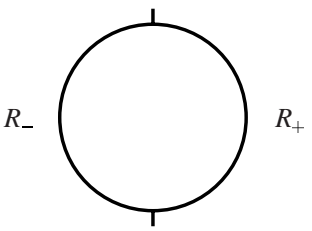
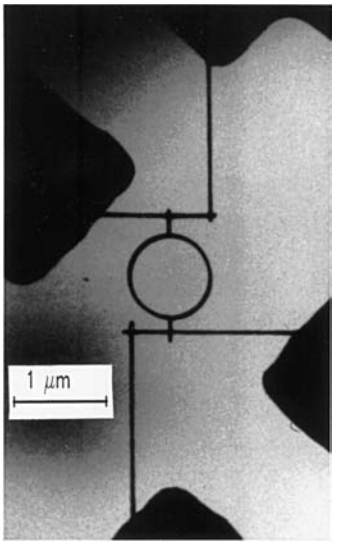


Figure 4.4: Two wires, with resistances R_+ and R_-, connected in parallel to two leads.

(a)



(b)

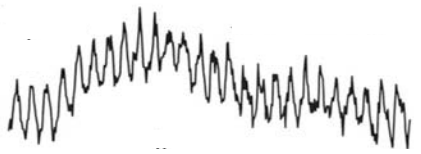


Figure 4.5: (a) Transmission electron micrograph of a ring of diameter 820 nm, made of gold wire 40 nm wide, from the experiment of Ref. 5. The leads allow joint measurements of potential difference and current. (Reprinted, with permission, from S. Washburn and R. A. Webb, *Adv. Phys.* **35** (1986) 375; C. P. Umbach fabricated the ring.) (b) Resistance oscillations, with period 75.9 gauss, measured on the gold ring. Multiplied by the area of the loop, 0.53 μm^2, the period is 4.0×10^{-7} maxwells (i.e. 4.0×10^{-7} gauss cm^2), agreeing well with $hc/e = 4.1 \times 10^{-7}$ maxwells.

prediction is exact if the flux does not touch the wires; and if the flux touches the wires, but the loop is much wider than the wires, we can still predict approximate periodicity.

The first demonstration of electron interference in a conducting loop was a 1985 experiment [5] on tiny gold loops (less than 1 micron in diameter) cooled to a temperature of 0.01 K (to minimize inelastic scattering). Figure 4.5(a) shows a photograph of one of the gold loops, and Fig. 4.5(b) shows the resistance of the loop as a function of magnetic field. The amplitude of the oscillations was only 0.1% of the total resistance, but the period of the oscillations times the area of the loop yielded hc/e to within the accuracy of the area measurement. This landmark demonstration of quantum phases at "mesoscopic" distances (some 10,000 times larger than atomic distances) was an application of the Aharonov–Bohm effect.

4.8 Non-Abelian Phases

In 1954, Yang and Mills [6] generalized the way gauges and phases transform together in the Schrödinger equation. Their generalization is an important part of elementary particle physics and other branches of physics. In the Yang-Mills construction, a wave function is a complex vector defined at each point in spacetime (rather than a complex number). We can write the

wave function as a column vector with N entries, each a complex function of $\mathbf{x}$ and t:

$$\mathbf{\Psi}(\mathbf{x},t) = \begin{pmatrix} \Psi_1(\mathbf{x},t) \\ \Psi_2(\mathbf{x},t) \\ \Psi_3(\mathbf{x},t) \\ \cdots \\ \Psi_N(\mathbf{x},t) \end{pmatrix} . \tag{4.19}$$

While before we considered multiplying $\Psi(\mathbf{x},t)$ by an arbitrary phase, now we may consider multiplying $\mathbf{\Psi}(\mathbf{x},t)$ by an arbitrary *unitary matrix*. Why unitary? The modulus of the complex wave function, $|\Psi(\mathbf{x},t)|$, is invariant under a change of phase. The norm of the complex vector Eq. (4.19), namely the square root of

$$|\mathbf{\Psi}(\mathbf{x},t)|^2 \equiv |\Psi_1(\mathbf{x},t)|^2 + |\Psi_2(\mathbf{x},t)|^2 + \cdots + |\Psi_N(\mathbf{x},t)|^2 ,$$

is invariant under multiplication of $\mathbf{\Psi}(\mathbf{x},t)$ by an arbitrary unitary matrix U: $|U\mathbf{\Psi}(\mathbf{x},t)| = |\mathbf{\Psi}(\mathbf{x},t)|$.

The vector and scalar potentials appearing in the Schrödinger equation will now be matrices rather than complex numbers. Like all operators representing quantum observables, they will be Hermitian matrices. Let the transformed wave function be $\mathbf{\Psi}'(\mathbf{x},t) = U\mathbf{\Psi}(\mathbf{x},t)$, where $\mathbf{\Psi}(\mathbf{x},t)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\mathbf{\Psi}(\mathbf{x},t) = \left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 \mathbf{\Psi}(\mathbf{x},t) + e\mathbf{V}(\mathbf{x})\mathbf{\Psi}(\mathbf{x},t) .$$

Then

$$i\hbar\frac{\partial}{\partial t}U^{-1}\mathbf{\Psi}'(\mathbf{x},t) = \left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 U^{-1}\mathbf{\Psi}'(\mathbf{x},t) + e\mathbf{V}(\mathbf{x})U^{-1}\mathbf{\Psi}'(\mathbf{x},t) .$$

Multiplying both sides by U, we obtain, after a little algebra,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\mathbf{\Psi}'(\mathbf{x},t) = (-i\hbar\nabla - \frac{e}{c}U\mathbf{A}U^{-1} - i\hbar U\nabla U^{-1})^2\mathbf{\Psi}'(\mathbf{x},t) \\ + eU\mathbf{V}U^{-1}\mathbf{\Psi}'(\mathbf{x},t) - i\hbar U\frac{\partial}{\partial t}U^{-1}\mathbf{\Psi}'(\mathbf{x},t) \end{aligned}$$

as the Schrödinger equation satisfied by $\mathbf{\Psi}'(\mathbf{x},t)$. Here, too, a transformation in the potentials $\mathbf{A}$ and V goes together with a transformation $\mathbf{\Psi}(\mathbf{x},t) \to \mathbf{\Psi}'(\mathbf{x},t)$; but this time, the gauge transformation is

$$\mathbf{A} \to U\mathbf{A}U^{-1} + i\frac{\hbar c}{e}U\nabla U^{-1} , \quad \mathbf{V} \to U\mathbf{V}U^{-1} - i\frac{\hbar}{e}U\frac{\partial}{\partial t}U^{-1} . \tag{4.20}$$

The gauge transformations Eqs. (4.5) and (4.20) are similar, since every unitary matrix U can be written as the exponential of a Hermitian matrix λ times i:

$$U = e^{i\lambda} .$$

Written this way, U is the direct generalization of the phase transformation $e^{i\lambda(\mathbf{x},t)}$ in Eq. (4.3). Indeed, $\lambda(\mathbf{x},t)$ in Eq. (4.3) can be regarded as a 1-dimensional Hermitian matrix, so that the transformations Eq. (4.5) are the simplest version of Eq. (4.20). The field $\mathbf{A}$ and others constructed from them are known as Yang-Mills fields. Since Hermitian matrices do not commute, in general, fields that transform according to the generalized gauge transformations Eq. (4.20) are also called *non-abelian* gauge fields, after the mathematician Abel, who studied commutative algebras. Now two wave packets of a particle can differ by a relative non-abelian phase, even if the non-abelian gauge field vanishes everywhere the wave packets are nonzero; that is, a non-abelian gauge field can produce a non-abelian Aharonov–Bohm effect [7]. Non-abelian gauge fields appear in nature; fields transforming according to Eq. (4.20) and built out of 2×2 and 3×3 matrices describe, respectively, the electroweak and the strong nuclear forces. And Sect. 12.2 shows how non-abelian phases can arise even in simple quantum systems.

Problems

4.1 Consider the rotating cylinder in Sect. 4.1 centered at the origin, with its axis along the z-axis. An electron moves in the xy-plane with coordinates $(x_0, vt, 0)$ so that its closest approach to the origin is $x_0 \gg r_c$. Use the law of Biot and Savart to show that $B_z(0,0,z,t)$, the z-component of the magnetic field at the point $(0,0,z)$ at time t, is

$$B_z(0,0,z,t) = -\frac{vx_0 e}{c(x_0^2 + v^2t^2 + z^2)^{3/2}} \ .$$

Use Faraday's law to show that the resulting torque on the cylinder is

$$I\frac{d^2\phi}{dt^2} = -\frac{Q{r_c}^2}{2ch}\int_{-l/2}^{l/2} dz \frac{\partial}{\partial t} B_z(0,0,z,t)$$

and obtain Eq. (4.1) in the approximation $x_0 \ll l$.

*4.2 It is possible to rewrite the Schrödinger equation, Eq. (4.8), so that only the gauge-invariant probability density ρ

$$\rho(\mathbf{x},t) \equiv |\Psi(\mathbf{x},t)|^2$$

and probability current $\mathbf{J}$

$$\mathbf{J} \equiv \frac{1}{2m}\Psi^*(-i\hbar\nabla - \frac{e}{c}\mathbf{A})\Psi + \frac{1}{2m}\Psi(i\hbar\nabla - \frac{e}{c}\mathbf{A})\Psi^*$$

appear [8]. (a) Show that Eq. (4.8) implies these two equations:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \ , \tag{4.21}$$

and

$$\frac{\partial}{\partial t}\left(\frac{m\mathbf{J}}{\rho}\right) = \nabla\left[\frac{\hbar^2}{4m\rho}\nabla^2\rho - \frac{\hbar^2}{8m\rho^2}(\nabla\rho)^2 - \frac{m\mathbf{J}^2}{2\rho^2}\right] + e\mathbf{E} \ . \tag{4.22}$$

(b) Show that $\mathbf{B} = -\nabla \times (mc\mathbf{J}/\rho e)$.
(c) Apparently, we have transformed the Schrödinger equation into two gauge-invariant equations, Eqs. (4.21–22). Now consider an initial wave function $\Psi(\mathbf{x}, 0)$ that vanishes for all $\mathbf{x}$ outside two disjoint regions. Show that $\rho(\mathbf{x}, 0)$, $\mathbf{J}(\mathbf{x}, 0)$ and Eqs. (4.21–22) do not determine $\Psi(\mathbf{x}, t)$ for all times $t > 0$.

4.3 (a) Consider the gauge choices $A_x = 0 = A_z$, $A_y = Bx$ and $A_y = 0 = A_z$, $A_x = -By$. Show that both correspond to a constant magnetic field of strength B along the z-axis, and that canonical momentum $\mathbf{p} = m\mathbf{v} - e\mathbf{A}/c$ depends on the choice of gauge.
(b) Show that Hamilton's equation of motion for Eq. (4.7) imply Eq. (4.6), and that the quantities appearing in Eq. (4.6) are measurable and gauge invariant.

4.4 Prove that

$$\frac{d}{dt}\langle\Psi|A|\Psi\rangle = \frac{i}{\hbar}\langle\Psi|[H, A]|\Psi\rangle + \langle\Psi|\frac{\partial A}{\partial t}|\Psi\rangle \ ,$$

where $|\Psi\rangle$ satisfies the Schrödinger equation, $i\hbar\partial|\Psi\rangle/\partial t = H|\Psi\rangle$. (If A does not depend explicitly on time, $\partial A/\partial t$ vanishes.) Take $H = p^2/2m + V(x, t)$ and prove Ehrenfest's theorem:

$$\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m} \ , \quad \frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle \ ,$$

where $\langle A\rangle \equiv \langle\Psi|A|\Psi\rangle$. Thus expectation values follow classical equations of motion.

4.5 Show that $\mathbf{A}(\mathbf{x}, t) = \Phi\nabla\theta/2\pi$ is an appropriate vector potential for the region outside a solenoid or a rotating, charged cylinder, where Φ is the enclosed flux and $\theta = \arctan\ (y/x)$.

4.6 Find a Lorentz-invariant expression for the Aharonov–Bohm phase that reduces to the scalar (electric) Aharonov–Bohm phase when the vector potential vanishes, and to the vector (magnetic) Aharonov–Bohm phase when the scalar potential vanishes.

4.7 Show that the phase factor $e^{ie\Lambda/\hbar c}$ in the Aharonov–Bohm effect depends on Λ only through its modular part Λ mod hc/e, and likewise $e^{ie\Phi_B/\hbar c}$ depends on Φ_B only through Φ_B mod hc/e. (By definition, X mod $hc/e = X - nhc/e$ for some integer n, and $0 \le X$ mod $hc/e < hc/e$.)

4.8 Applying the definition $L_z = xp_y - yp_x$ to the operators $\mathbf{x}$ and $\mathbf{p} = -i\hbar\nabla$, derive the operator $L_z = -i\hbar\partial/\partial\phi$ in spherical coordinates.

*4.9 Both $p \to -i\hbar\partial/\partial x$ and $x \to i\hbar\partial/\partial p$ represent the commutation relation $[x, p] = i\hbar$. If in the former case the Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\Psi(x, t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x, t) \ ,$$

where $V(x)$ is a polynomial in x, in the latter case it is

$$i\hbar\frac{\partial}{\partial t}\tilde{\Psi}(p,t) = \left[\frac{p^2}{2m} + V\left(i\hbar\frac{\partial}{\partial p}\right)\right]\tilde{\Psi}(p,t)\ ,$$

where $\tilde{\Psi}(p,t)$ is the Fourier transform of $\Psi(x,t)$. (a) How must $i\hbar\partial/\partial p$ transform for the latter Schrödinger equation to be invariant under multiplication of $\tilde{\Psi}(p,t)$ by a p-dependent phase?
(b) What physical principle would this invariance violate?

4.10 The field $\mathbf{F}_{jk}$ corresponding to a vector potential $\mathbf{A}$ is

$$\mathbf{F}_{jk} = \partial_j\mathbf{A}_k - \partial_k\mathbf{A}_j - \frac{ie}{\hbar c}[\mathbf{A}_j, \mathbf{A}_k]\ .$$

(In electromagnetism, $\mathbf{A}$ is abelian and the commutator vanishes.) Show that $\mathbf{F}$ is gauge covariant, i.e. that the gauge transformation Eq. (4.20) implies

$$\mathbf{F}_{jk} \to U\mathbf{F}_{jk}U^{-1}\ .$$

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5 Modular Variables

Previous chapters show a close relationship between quantum interference and the uncertainty principle. The uncertainty principle implies loss of quantum interference whenever we can detect through which slit a particle passes. (See Sects. 2.4 and 4.5.) Nevertheless, the relationship between interference and the uncertainty principle remains obscure. For example, consider partial quantum waves (of a single particle) emerging from two slits. The partial waves do not yet overlap as they emerge from the slits. If we detect through which slit the particle passes, we decrease the uncertainty in the particle's transverse position. A natural guess, based on the uncertainty relation, Eq. (2.5), is that we increase the uncertainty in the particle's transverse momentum. But a simple calculation (Prob. 5.1) shows that the uncertainty in the transverse momentum does not change. Why, then, does the interference pattern disappear? We have analyzed several detection schemes and each time applied the uncertainty principle to the *detector* to explain the loss of interference; but applying the uncertainty principle to the *particle* tells us nothing about loss of interference.

Another example of the obscure relation between interference and the uncertainty principle is the Heisenberg formulation of quantum mechanics. The Heisenberg formalism (in which operators evolve in time and states do not) is equivalent to the Schrödinger formalism; the Heisenberg uncertainty principle falls naturally out of both formulations. Yet how can we think about interference in the Heisenberg formalism? Without wave packets that evolve in time, it seems impossible.

These examples suggest that we have not learned to think intuitively about interference. Relativity theory introduced an intuitive language for describing space and time, but in quantum theory we still describe interference as if the interfering waves were classical. This chapter introduces a new language for describing quantum interference. We begin with a paradox involving a lattice of solenoids. The solution to this paradox will give us new intuition about both interference and quantum nonlocality.

5.1 A Lattice of Solenoids

Consider electrons of wavelength λ diffracting through a heavy grating, as shown in Fig. 5.1; the grating consists of narrow slits, spaced a distance L apart, formed by parallel slats. The electrons scatter into discrete directions defined by angles θ_n (n an integer):

$$\sin\theta_n = n\lambda/L \ ;$$

in these directions the partial electron waves interfere constructively. Because the grating is heavy, the energy of the electrons is practically the same before and after diffraction; hence their

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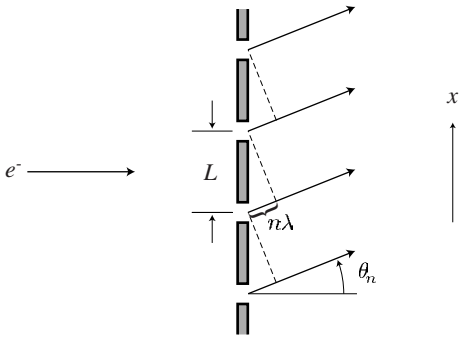

Figure 5.1: Electron diffracting through a grating at an angle θ_n where $\sin\theta_n = n\lambda/L$. Electron paths differing in length by $n\lambda$ interfere constructively.

momentum $p = h/\lambda$ and wavelength remain the same, according to the de Broglie relation, Eq. (2.4). If the incident electrons move parallel to the y-axis and diffract in the xy-plane, the transverse momentum p_x of an electron scattered through an angle θ_n is

$$p_x = p\sin\theta_n = nh/L\ . \tag{5.1}$$

Assuming the grating is free to move in the x-direction, the x-component of momentum is conserved during diffraction, and the grating acquires momentum nh/L in the x-direction from the electron. Note that the electron and grating exchange transverse momentum only in multiples of h/L. We can derive this constraint also from a perturbative calculation. Let Ψ and Ψ' be initial and final electron states of momenta $\hbar k_x$ and $\hbar k'_x$, respectively. To first order in the perturbation, the transition probability from Ψ to Ψ' is proportional to

$$\langle\Psi'|V|\Psi\rangle \simeq \int_{-\infty}^{\infty} e^{i(k_x - k'_x)x} V(x)dx\ ,$$

where $V(x)$ is the perturbing potential. This integral is just the Fourier transform of $V(x)$ with respect to x. If (like the grating) $V(x)$ is periodic in x with period L, the integral vanishes unless $\hbar(k'_x - k_x) = nh/L$.

Let us now modify the experiment [1]. Suppose that each slat in the grating is hollow, and inside each slat we place a solenoid. The solenoids are rigidly connected to one another, forming a lattice. (See Fig. 5.2.) Imagine the ends of each slat open and all the solenoids connected above and below by a rod, while short plates connect the slats above and below. Yet the solenoids are not connected to the grating, so the lattice of solenoids moves independently of the grating (except for the constraint that the solenoids must stay inside the slats). As we saw in Sect. 4.4, a solenoid carrying a flux Φ_B contributes $e\Phi_B/\hbar c$ to the relative phase of partial waves passing on either side of it. If all the solenoids carry the same flux Φ_B, then electrons, after passing the grating, will scatter into a new set of angles θ'_n defined by

$$\sin\theta'_n = \left(n + \frac{e\Phi_B}{2\pi\hbar c}\right)\lambda/L\ . \tag{5.2}$$

For example, if the extra phase due to the solenoids is $e\Phi_B/\hbar c = \pi$, then the pattern of lines of constructive interference will be shifted by half the separation between neighboring lines.

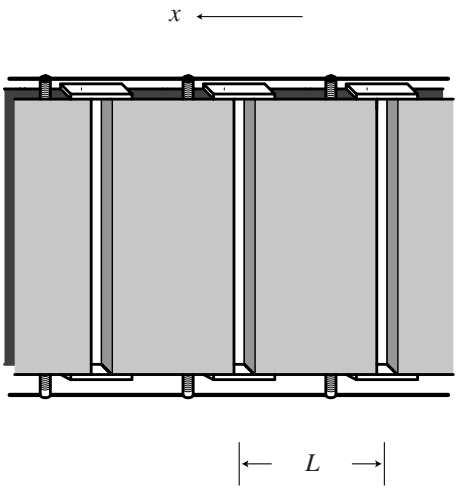

Figure 5.2: A lattice of solenoids confined by a grating.

Constructive interference now corresponds to a change in the electrons' transverse momentum of

$$p_x = p \sin\theta'_n = \left(n + \frac{1}{2}\right) h/L \; .$$

(See Fig. 5.3.) Since diffraction of the electrons conserves total transverse momentum, we might assume that the grating, again, picks up the transverse momentum lost by the electrons. But, once again, the electrons and grating exchange transverse momentum only in multiples of h/L. We obtained this constraint from a perturbative calculation with the same interaction between the electrons and the grating. Therefore, the *solenoids* must absorb $\pm h/2L$ of the transverse momentum carried by the electrons, even though there is never any contact between the solenoids and the electrons. We can be sure that there is no contact between the solenoids and the electrons, because we have placed the solenoids within the hollow slats of the grating.

Here, as in Sect. 4.1, we can arrange for the electric and magnetic fields of each solenoid to vanish wherever the electrons go. Thus the electrons and solenoids must exchange momentum *nonlocally*. A startling effect! But could we ever observe this effect? The fact that the solenoids

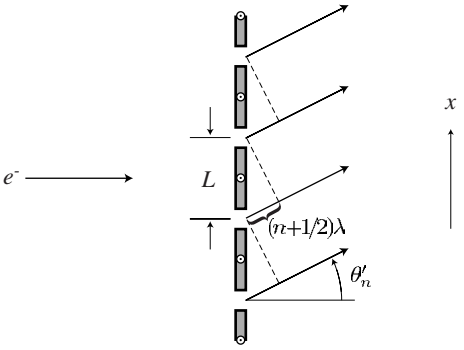

Figure 5.3: Electrons diffracting through the grating and lattice of solenoids of Fig. 5.2 at angles θ'_n where $\sin\theta'_n = (n + 1/2)\lambda/L$. Each solenoid contains flux $hc/2e$; thus electron paths differing in length by $(n + 1/2)\lambda$ interfere constructively.

must remain inside the slats implies that the uncertainty in their transverse position is less than L. For Δx strictly less than or equal to L, the uncertainty in p_x is $\Delta p_x \geq h/L$. (See Prob. 5.3.) Consequently, the uncertainty in their transverse momentum cannot be less than h/L, and there is no possibility of detecting a change in the solenoid momentum of $\pm h/2L$. Nonlocal exchange of momentum is apparently an unobservable (hence less than startling) effect.

Yet there is something wrong with this argument. If we send one electron through the grating, the transverse momentum of the solenoids must change by at least $h/2L$ in absolute magnitude; suppose it does not change by more than that. So the solenoids pick up transverse momentum $\pm h/2L$. We cannot detect such a small exchange. But let us send another electron through the grating. This electron interacts with the grating and the solenoids in the same way; once again, the solenoids pick up transverse momentum $\pm h/2L$. We send in a third electron; the solenoids pick up another $\pm h/2L$ of transverse momentum. There is no reason why all these contributions to the transverse momentum of the solenoids should add with the same sign. But there is no reason why they should all exactly cancel. Indeed, the contributions of the electrons to the transverse momentum should be like steps in a one-dimensional random walk: each step is equal in size but random in direction, like the steps of a drunk who has forgotten where he is going. After N such steps the net displacement will most likely be $\pm\sqrt{N}$ times the step size. Thus, after we send N electrons through the grating, the solenoids are likely to pick up $\pm\sqrt{N}h/2L$ (or more) in transverse momentum! The uncertainty principle cannot prevent us from observing this nonlocal exchange of momentum, which can be arbitrarily large.

The effect is not only startling, it is paradoxical. It contradicts both experiment and theory. It is paradoxical for solenoids to accelerate without any forces acting on them; if such an effect exists in quantum physics, then the correspondence principle implies it should exist in classical physics as well.[1] We can ignore this paradox if we cannot observe nonlocal exchanges of momentum, but we cannot ignore what we can observe. If nonlocal exchanges of momentum between electrons and solenoids accumulate like steps in a random walk, then they must be observable, because a random walk diverges. We might question whether the electrons and grating can indeed exchange transverse momentum only in multiples of h/L, since we obtained this constraint by treating the grating to first order as a perturbation. However, there is a simple, exact proof of the constraint. (See Prob. 5.5.)

We close this section with a hint towards solving this paradox. A random walk on a line diverges. A random walk on a circle does not. No point on a circle is farther than half the circumference from any other point. This observation may seem irrelevant, because electrons and solenoids exchange momentum, an unbounded quantity, while a circle corresponds to a bounded and periodic quantity, such as an angle. Yet (from Sect. 4.4) what counts in the Aharonov–Bohm effect is the *modular* part of the phase, e.g. $e\Phi_B/\hbar c$ mod 2π. (See Prob. 4.7.) If the solenoids in the lattice all carry flux Φ_B, the transverse momentum of an electron scattered through an angle θ'_n is (from Eq. (5.2))

$$p_x = p\sin\theta'_n = \left(n + \frac{e\Phi_B}{2\pi\hbar c}\right) h/L \ .$$

[1] According to Bohr's correspondence principle, the predictions of quantum physics and classical physics must correspond in the limit of many quanta.

Since the grating can pick up transverse momentum in multiples of h/L only, the lattice of solenoids must absorb some transverse momentum, too. All we can determine, however, is the *modular* part of the transverse momentum they absorb, i.e. $(n + e\Phi_B/2\pi\hbar c)\,(h/L)$ mod h/L. How does this quantity accumulate as the electrons diffract?

5.2 Non-overlapping Wave Packets

Section 4.2 discusses interference between two partial quantum waves $\Psi_1(\mathbf{x}, t)$ and $\Psi_2(\mathbf{x}, t)$. It states that the interference pattern is invariant under a local phase transformation $e^{i\lambda(\mathbf{x},t)}$ because, at a given spacetime point $(\mathbf{x}, t)$, the same factor $e^{i\lambda(\mathbf{x},t)}$ multiplies $\Psi_1(\mathbf{x}, t)$ and $\Psi_2(\mathbf{x}, t)$. This statement assumes (implicitly) that interference requires the two partial waves to meet or overlap at a spacetime point. Let us look closer at this assumption.

A partial wave emerging from a slit is localized in space. We can describe such a localized wave as a *wave packet*, i.e. a superposition of waves of different wavelengths. Two kinds of wave packets are of special interest. One is the standard minimal uncertainty wave packet (in one space dimension), the gaussian function

$$\psi_g(x) = \frac{e^{-x^2/2a^2}e^{ikx}}{(\pi a^2)^{1/4}}\ , \tag{5.3}$$

peaked at $x = 0$, for which $\Delta x = a/\sqrt{2}$ and $\Delta p = \hbar/\sqrt{2}a$. (See Prob. 3.10 for the definition of Δx and Δp.) This wave packet does not vanish for any x, but it decays exponentially in x. It is relatively easy to handle mathematically; the Schrödinger equation for a free particle of mass m on a line,

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)$$

has an exact solution coinciding with $\psi_g(x)$ at $t = 0$:

$$\Psi(x,t) = \frac{e^{-i\hbar k^2 t/2m + ikx}e^{-(x-\hbar kt/m)^2/2(a^2+i\hbar t/m)}}{\pi^{1/4}\,(a + i\hbar t/ma)^{1/2}}\ . \tag{5.4}$$

Equation (5.4) describes a wave packet peaked at $x = \hbar kt/m$; i.e. the wave travels at average speed $p/m = \hbar k/m$. Another kind of wave packet is the test function $\psi_t(x)$; it equals

$$\psi_t(x) = \begin{cases} e^{-1/(a^2-x^2)} & \text{if } |x| \le a; \\ 0 & \text{if } |x| \ge a, \end{cases} \tag{5.5}$$

up to normalization. (See Fig. 5.4.)

Like the minimal uncertainty wave packet, the test function is continuous, and has continuous derivatives of all orders. It has a well-defined Fourier transform. However, since its derivatives vanish for $|x| \ge a$, it does not have a Taylor expansion there; unlike the minimal uncertainty wave packet, it is not analytic. (As a function of complex x, it has essential singularities.) Non-analytic functions are worth considering. Indeed, we cannot describe the

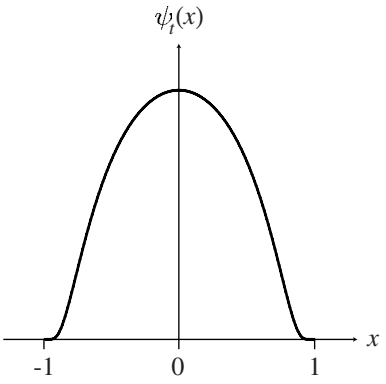

Figure 5.4: The test function of Eq. (5.5), with $a = 1$.

world without them. Relativistic causality states that a cause at one spacetime point can have no measurable effect at another spacetime point, until a time equal to their spatial separation divided by the speed of light c has elapsed. If we change a quantum wave at one spacetime point, the rest of the wave cannot change instantly, so the entire wave function cannot have a Taylor expansion. Thus relativistic causality is incompatible with complete analyticity.

Now consider two wave packets, ψ_1 and ψ_2, on the x-axis. With these two wave packets let us define (up to normalization) a class of wave functions Ψ_α,

$$\Psi_\alpha = \psi_1 + e^{i\alpha}\psi_2 \ ,$$

where α is a constant. The overall phase is unmeasurable, thus $\psi_1 + e^{i\alpha}\psi_2$ and $e^{-i\alpha}\psi_1 + \psi_2$ are equivalent. But if two phase factors $e^{i\alpha}$ and $e^{i\beta}$ are different, then Ψ_α and Ψ_β are physically different. If ψ_1 and ψ_2 overlap and interfere (and they will as Ψ_α evolves) the resulting interference pattern depends on the relative phase α. Nevertheless, as long as they do not overlap, the relative phase α affects neither the probability density nor the current density. Of course, if ψ_1 and ψ_2 are minimal uncertainty wave packets, they always overlap to some extent. Strictly speaking, two wave packets ψ_1 and ψ_2 do not overlap only if they have no common support. For example, let ψ_1 be the test function of Eq. (5.5) centered at $x = 0$ and let ψ_2 be the test function centered at $x = L$ with $L > 2a$. (See Fig. 5.5.) Then ψ_1 and ψ_2 are strictly non-overlapping.

We now prove a fundamental theorem about interference: If $\Psi_\alpha(x, 0) = \psi_1 + e^{i\alpha}\psi_2$

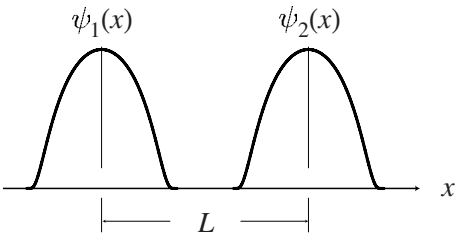

Figure 5.5: Two non-overlapping wave packets on the x-axis.

evolves freely and $\psi_1(x)$ and $\psi_2(x)$ do not overlap, then

$$\int_{-\infty}^{\infty} \Psi_\alpha^*(x,t) x^j p^k \Psi_\alpha(x,t) dx \tag{5.6}$$

is independent of α for all t and for all j and k. In Eq. (5.6), p is the operator $-i\hbar\partial/\partial x$; thus ψ_1^* and $x^j p^k \psi_2$ do not overlap, and neither do ψ_2^* and $x^j p^k \psi_1$. We have proved the theorem for $t = 0$, because if ψ_1 and ψ_2 do not overlap, then $\Psi_\alpha^* \Psi_\alpha$ does not depend on α, and neither does $\Psi_\alpha^* x^j p^k \Psi_\alpha$. To extend the theorem to all t, we define a unitary operator $U(t)$ such that

$$\Psi_\alpha(x,t) = U(t)\Psi_\alpha(x,0) = e^{-ip^2 t/2m\hbar}\Psi_\alpha(x,0) \; ;$$

then $\Psi_\alpha(x,t)$ solves the free Schrödinger equation and the integrand is

$$\begin{aligned} \Psi_\alpha^*(x,t) x^j p^k \Psi_\alpha(x,t) &= \Psi_\alpha^*(x,0) U^\dagger(t) x^j p^k U(t) \Psi_\alpha(x,0) \\ &= \Psi_\alpha^*(x,0) [U^\dagger(t) x U(t)]^j [U^\dagger(t) p U(t)]^k \Psi_\alpha(x,0) \\ &= \Psi_\alpha^*(x,0) [x + pt/m]^j p^k \Psi_\alpha(x,0) \end{aligned}$$

which again does not depend on α.

5.3 Modular Momentum

Interference of two wave packets ψ_1 and ψ_2 reveals their relative phase α, but the wave packets must overlap to interfere, as Sect. 4.2 assumes. And from the theorem in the last section, we learn that the expectation values of operators of the form $x^j p^k$ depend on the relative phase only if the wave packets overlap. It looks as if no physical observable reveals the relative phase, as long as the wave packets ψ_1 and ψ_2 do not overlap. Yet consider the operator $e^{ipL/\hbar}$. In general,

$$e^{ipL/\hbar} f(x) = f(x + L) \tag{5.7}$$

for any function $f(x)$ with a Fourier transform. If Ψ_α consists of two wave packets separated by a distance L, then $e^{ipL/\hbar}$ brings them together. For example, if Ψ_α is the normalized sum of $\psi_t(x)$ and $e^{i\alpha}\psi_t(x - L)$, then the expectation value $\langle \Psi_\alpha | e^{ipL/\hbar} | \Psi_\alpha \rangle$ is simply $e^{i\alpha}/2$. While $e^{ipL/\hbar}$ is not a self-adjoint operator, $e^{ipL/\hbar} + e^{-ipL/\hbar}$ is; its expectation value is $\cos\alpha$.

Clearly, $\cos\alpha$ depends on α. But suppose we expand $e^{ipL/\hbar} + e^{-ipL/\hbar} = 2\cos(pL/\hbar)$ in powers of p: we have

$$\cos(pL/\hbar) = 1 - \frac{(pL/\hbar)^2}{2!} + \frac{(pL/\hbar)^4}{4!} - \dots \; .$$

We might claim

$$\langle \cos(pL/\hbar) \rangle = 1 - \frac{\langle p^2 \rangle (L/\hbar)^2}{2!} + \frac{\langle p^4 \rangle (L/\hbar)^4}{4!} - \dots \; ,$$

but each term in this series is independent of α, so the sum should be independent of α. Try to explain this contradiction before reading on!

To prove that Eq. (5.7) holds for any function $f(x)$ with a Fourier transform $\tilde{f}(p)$, we write

$$f(x) = (2\pi\hbar)^{-1/2} \int \tilde{f}(p) e^{ipx/\hbar} dp \; ;$$

then

$$\begin{aligned} e^{ipL/\hbar} f(x) &= e^{ipL/\hbar} (2\pi\hbar)^{-1/2} \int \tilde{f}(p) e^{ipx/\hbar} dp \\ &= (2\pi\hbar)^{-1/2} \int \tilde{f}(p) e^{ip(x+L)/\hbar} dp \\ &= f(x+L) \; . \end{aligned}$$

Note that p is an operator only outside the integral. On the other hand, if we expand

$$\begin{aligned} e^{ipL/\hbar} f(x) &= \sum_{n=0}^{\infty} \frac{(ipL/\hbar)^n}{n!} f(x) \\ &= \sum_{n=0}^{\infty} \frac{L^n}{n!} \frac{d^n}{dx^n} f(x) \; , \end{aligned} \tag{5.8}$$

we get a series which sums to $f(x+L)$ if and only if $f(x)$ is analytic, i.e. if the second line in Eq. (5.8) is the Taylor series expansion of $f(x+L)$. Equation (5.8) holds only for analytic functions, but Eq. (5.7) holds for all functions having a Fourier transform. Not all quantum wave functions are analytic, but all quantum wave functions have a Fourier transform (momentum representation).

The operator $e^{ipL/\hbar}$ reveals the relative phase α, a nonlocal aspect of Ψ_α, because it translates the wave function. By contrast, operators of the form $x^j p^k$ reveal nothing about α until the wave packets overlap and interfere. We see now that the usual statement of the uncertainty principle, $\Delta x \Delta p \geq \hbar/2$ with $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$, could tell us nothing about interference, because $\langle x \rangle$, $\langle p \rangle$, Δx and Δp are all independent of the relative phase. The fact that $\langle \cos(pL/\hbar) \rangle$ and other observables *do* depend on the relative phase suggests that we can find uncertainty relations that apply directly to quantum interference. To find them, we start by asking what information $e^{ipL/\hbar}$ contains. If we replace p by $p - nh/L$ in $e^{ipL/\hbar}$, the operator remains invariant, since $e^{in2\pi} = 1$. So $e^{ipL/\hbar}$ does not depend on all of p; it depends only on p mod nh/L. We call this quantity the *modular* momentum and denote it p_{mod}. Thus p_{mod} is defined relative to L and takes values in the interval $[0, h/L)$. If we know something about p_{mod}, then some of these values are more probable than others; if we know nothing about p_{mod}, then all the values are equally probable and the expectation value

$$\langle e^{ip_{mod}L/\hbar} \rangle = \langle e^{ipL/\hbar} \rangle = 0$$

vanishes because the values of $p_{mod}L/\hbar$ are distributed with equal probability in the interval $[0, 2\pi)$. The converse is not true; for example, $\langle e^{ipL/\hbar} \rangle$ vanishes if $p_{mod} = 0$ and $p_{mod} = h/2L$ both have probability $1/2$. But if $\langle e^{inpL/\hbar} \rangle$ vanishes for every positive integer n, then all the values of p_{mod} are equally probable.[2] We state this result as the *complete uncertainty* principle:

[2] We can think of $p_{mod}L/\hbar$ as an angle, since it takes values in the interval $[0, 2\pi)$. Then $\langle e^{inpL/\hbar} \rangle$ is the n-th Fourier component of the distribution of this angle. If $\langle e^{ipL/\hbar} \rangle$ vanishes for all positive n, then it vanishes for all negative n (since the distribution is real) and the distribution is flat.

p_{mod} is completely uncertain if and only if $\langle e^{inpL/\hbar}\rangle$ vanishes for every positive integer n. Note that $\langle e^{inpL/\hbar}\rangle$ vanishes in any state $\Psi(x)$ confined to an interval of length L, for then $\Psi(x)$ and $e^{inpL/\hbar}\Psi(x)$ have no common support.

This result directly explains the connection between uncertainty and the breakdown of interference. Let ψ_1 and ψ_2 represent wave packets emerging from two slits separated by a distance L, with relative phase α. In the state Ψ_α the expectation value of $e^{ipL/\hbar}$ does *not* vanish; in either of the states ψ_1 or ψ_2 it *does*. Thus what is lost when we detect though which slit a particle passes (and reduce the state Ψ_α to either ψ_1 or ψ_2) is *all knowledge of the modular momentum* p_{mod}.

5.4 The x_{mod}, p_{mod} Representation

Having defined p_{mod}, we may look for an analogous quantity for x. For a grating of slits spaced a distance L apart, the modular momentum of an electron is $p_{mod} = p \bmod h/L$ because the electron and grating can exchange transverse momentum only in multiples of h/L. The electron passes this grating via the slits, but we do not know which slit. What we know is the electron's transverse position modulo L. So we define the modular position to be $x_{mod} = x \bmod L$. We have the identity

$$e^{i2\pi x_{mod}/L} = e^{i2\pi x/L}$$

analogous to the identity $e^{ip_{mod}L/\hbar} = e^{ipL/\hbar}$. We could, of course, define $x_{mod} = x \bmod L'$ for arbitrary L', but $L' = L$ fits the physical conditions. The fact that p_{mod} and x_{mod} are mutually defined makes them satisfy a particularly simple commutation relation. We can calculate the commutator of $e^{ip_{mod}L/\hbar}$ and $e^{i2\pi x_{mod}/L'}$ for arbitrary L' by applying Eq. (5.7). We have

$$e^{ipL/\hbar}e^{i2\pi x/L'}f(x) = e^{i2\pi(x+L)/L'}f(x+L) = e^{i2\pi x/L'+i2\pi L/L'}e^{ipL/\hbar}f(x)\ ,$$

for any function $f(x)$ with a Fourier transform; therefore p_{mod} and x_{mod} commute,

$$\left[e^{ip_{mod}L/\hbar}, e^{i2\pi x_{mod}/L'}\right] = 0\ , \tag{5.9}$$

if and only if L/L' is an integer [2]. In general, two operators $f(x)$ and $g(p)$ commute only if both are periodic and the product of their periods is a multiple of h. Figure 5.6 shows that x_{mod} is periodic in x with period L and p_{mod} is periodic in p with period h/L; the product of the periods is h. Unlike x and p, x_{mod} and p_{mod} commute.

Since x_{mod} and p_{mod} commute, no uncertainty principle keeps us from measuring them at the same time. We can represent our knowledge of x and p by dividing phase space into cells of area h as in Fig. 5.7. Figure 5.7 shows the case of $x_{mod} = L/4$ and $p_{mod} = h/2L$. They could represent, for example, the modular position and momentum (along the x-axis) of the electrons in the paradox of Sect. 5.1 (with solenoids shifting the phase by $e\Phi_B/\hbar c = \pi$). The electrons must pass through the slits in the grating; we only know each electron's transverse position modulo L, since we do not know through which slit it passed. The lattice of solenoids shifts the angles, and thus the transverse momenta, of the diffracted electrons; we only know their

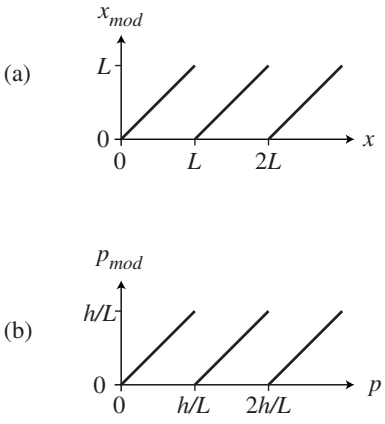

Figure 5.6: (a) Graph of x_{mod} as a function of x. (b) Graph of p_{mod} as a function of p.

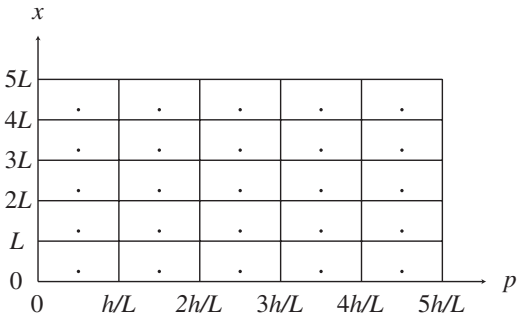

Figure 5.7: Phase-space representation of the state in which $x_{mod} = L/4$ and $p_{mod} = h/2L$.

transverse momenta modulo h/L, since we do not know through which angle each electron diffracts.

The fact that x_{mod} and p_{mod} commute has great significance. Quantum mechanics limits our knowledge in a way that classical physics does not, but not in the way we might expect. If $f(x)$ represents our knowledge of a particle's position x and $g(p)$ our knowledge of its momentum p, they are compatible only if $[f(x), g(p)] = 0$. Since x and p do not commute, we might conclude that knowing a particle's position is incompatible with knowing whether, say, its momentum is conserved. However, x_{mod} and p_{mod} provide a representation in which we can have exact information about both position and momentum. This representation is unique and exists for any conjugate pair of physical variables, including energy and time, allowing partial but exact information about both variables.

The paradox in Sect. 5.1 arises from the fact that the interaction among the electrons, the grating and the solenoids conserves momentum, and momentum is an unbounded quantity. Exchanges of momentum between the electrons and the solenoids, no matter how small, must eventually amount to a measurable effect. But now that we have defined modular momentum, we see that the interaction between the electrons and the solenoids separately conserves p_{mod}. If the sum of the momenta p_e of the electrons, p_{gr} of the grating and p_{sol} of the lattice of solenoids is conserved, we have

$$\cos([p_e + p_{gr} + p_{sol}]L/\hbar) = \text{constant} \; . \tag{5.10}$$

Since p_{gr} changes in multiples of h/L, the modular momentum of the grating does not change. Equation (5.10) then defines a relation between the modular momenta of the electrons and solenoids.[3] The electrons and solenoids exchange modular momentum via nonlocal phases; but from the theorem of Sect. 5.2 we infer that the expectation values $\langle p_e^2 \rangle$ and $\langle p_{sol}^2 \rangle$ do not depend on nonlocal phases. Hence the changes in momentum do not accumulate to a measurable effect. The changes in the modular momentum of the solenoids are like a random walk on a circle.

After defining x_{mod} and p_{mod} we define also N_x and N_p according to

$$x = N_x L + x_{mod} \ , \quad p = N_p (h/L) + p_{mod} \ .$$

Applying Eq. (5.9), we obtain the relations

$$\left[e^{i2\pi x_{mod}/L}, N_p \right] = -e^{i2\pi x_{mod}/L} \ , \quad \left[e^{ip_{mod}L/\hbar}, N_x \right] = e^{ip_{mod}L/\hbar} \ , \tag{5.11}$$

which recall the relation $\left[L_z, e^{i\phi} \right] = \hbar e^{i\phi}$; that is, x_{mod} and p_{mod} are analogous to angles. In the same way that L_z, ϕ form a conjugate pair, so do $\hbar N_x$ and $(L/\hbar)p_{mod}$, and also $\hbar N_p$ and $(2\pi/L)x_{mod}$; their uncertainties are related approximately by

$$\Delta N_x \Delta p_{mod} \geq h/L \ , \quad \Delta N_p \Delta x_{mod} \geq L \ . \tag{5.12}$$

The usual uncertainty principle states that if we try to localize a particle in phase space, the best we can do is localize it in a cell of area h. As Eq. (5.12) and Fig. 5.7 show, if we don't ask in which cell it is, we can know exactly where it is in a cell. When do we need x_{mod} and p_{mod}? The observables x and p and their uncertainty relation are adequate for analyzing a single wave packet. For a single wave packet we know N_x and N_p very well, so x_{mod} and p_{mod} are almost completely unknown. But to analyze interference and other phenomena involving more than one wave packet, we need modular variables.

Note the contrast between Eq. (5.9) and its classical counterpart, the Poisson bracket. If we require the Poisson bracket of two functions $f(x)$ and $g(p)$ to vanish, we have

$$0 = \{ f(x), g(p) \} = \frac{df}{dx} \frac{dg}{dp} \ . \tag{5.13}$$

Equation (5.13) implies that either $f(x)$ or $g(p)$ is a constant, so the only solutions are trivial. So what happens to x_{mod} and p_{mod} in the classical limit? Taking $\hbar \to 0$ we find that x_{mod} and p_{mod} cannot both have any definite value; thus nonlocal interference effects must disappear in the classical limit.

5.5 Intimations of Nonlocality

In the Schrödinger formulation of quantum mechanics, the unitary operator $U(t)$

$$U(t) = e^{-iHt/\hbar} \tag{5.14}$$

[3] The graph of $\cos(p_e L/\hbar)$ versus $\cos(p_{sol} L/\hbar)$ is an ellipse; see Prob. 5.10.

evolves a state $|\Psi\rangle$ forward in time:

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle \ . \tag{5.15}$$

Schrödinger's equation is the time derivative of Eq. (5.15). For simplicity, we have taken H to be time independent in Eq. (5.14). In the Schrödinger formalism any operator Ω_s, including H, is unchanging unless it depends explicitly on time. In the Heisenberg formalism, by contrast, states are unchanging and the operator Ω_h corresponding to Ω_s evolves in time according to

$$\Omega_h(t) = U^{-1}(t)\Omega_s U(t) \ . \tag{5.16}$$

Thus expectation values and eigenvalues come out the same in the two formulations. The time derivative of Eq. (5.16) gives us the equation of motion for Ω_h:

$$\frac{d\Omega_h}{dt} = \frac{i}{\hbar}[H, \Omega_h] + U^{-1}(t)\frac{\partial \Omega_s}{\partial t}U(t) \ . \tag{5.17}$$

Consider the Heisenberg equation of motion for the operator $e^{ipL/\hbar}$ when the Hamiltonian is $H = p^2/2m + V(x)$. Equation (5.17) yields

$$\frac{d}{dt}e^{ipL/\hbar} = \frac{i}{\hbar}[V(x) - V(x+L)]e^{ipL/\hbar} \ , \tag{5.18}$$

since $e^{ipL/\hbar}$ generates displacements. This difference equation involves the potential evaluated at x and at $x + L$. How does it compare with the classical equation of motion? The limit of $e^{ipL/\hbar}$ as $\hbar \to 0$ does not exist if L is constant; so let us rewrite $e^{ipL/\hbar}$ as $e^{i2\pi p/p_0}$ with $p_0 = h/L$, and keep p_0 constant. Then both L and $\hbar$ go to 0 in the classical limit. The classical equation of motion for $e^{i2\pi p/p_0}$ is

$$\frac{d}{dt}e^{i2\pi p/p_0} = \left\{e^{i2\pi p/p_0}, H\right\} = -i\frac{2\pi}{p_0}\frac{dV}{dx}e^{i2\pi p/p_0} \ , \tag{5.19}$$

and contains a derivative rather than a difference. Still, the two equations of motion are consistent: $[V(x) - V(x+L)]/\hbar$ in Eq. (5.18) equals $(2\pi/p_0 L)[V(x) - V(x+L)]$ and approaches $-(2\pi/p_0)dV/dx$ as $L \to 0$.

Yet there is a fundamental difference between the classical and quantum equations of motion. The classical equation of motion, Eq. (5.19), is *local*; $e^{i2\pi p/p_0}$ changes only if dV/dx is nonzero, i.e. if a force acts directly at x. The quantum equation of motion, Eq. (5.18), is *nonlocal*: $e^{i2\pi p/p_0}$ can change even if dV/dx is zero at the particle's position, but $V(x + L) - V(x) \neq 0$. Figure 5.8 shows a wave packet in a staircase potential $V(x) = V_0 N_x$. The wave packet, on the step in the interval $(L, 2L)$, is nonzero only where $V(x)$ is constant and there is no force. We don't expect the momentum of the particle to change. But the modular momentum *does* change: $V(x + L) - V(x) = V_0$ for all x, and so the modular momentum behaves as $p_{mod}(t) = [p_{mod}(0) - V_0 t/\hbar] \bmod h/L$, whatever the particle's state.

Because it obeys a nonlocal equation of motion, modular momentum is the key to describing interference in Heisenberg's formulation. Let's analyze, first of all, the interaction of an electron and a grating. The electron passes the grating, so we have precise information about x_{mod} (in

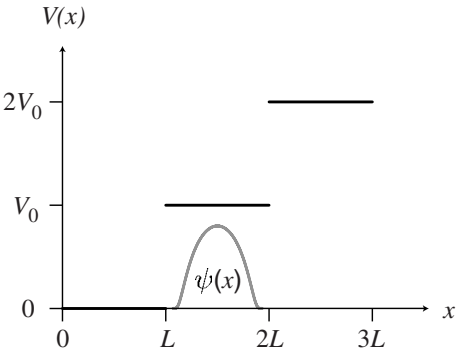

Figure 5.8: A staircase potential $V(x) = V_0 N_x$ and a wave packet with support only in the interval $L < x < 2L$.

the transverse direction); x_{mod} and p_{mod} commute, so we have precise information about p_{mod}. But x_{mod} and N_p do not commute (see Eqs. (5.11–12)) so we have no information about N_p. The interaction of the electron with the grating conserves p_{mod} (see Prob. 5.5) so these facts determine the interference pattern completely: p_{mod} fixes the position of the fringes relative to the grating; N_p is completely uncertain and therefore the fringes are equally dense. Now consider the effect of a lattice of solenoids. The solenoids affect the modular momentum in the same way as the stair potential $V(x)$ of Fig. 5.8. The nonlocal interaction of the electrons with the solenoids changes p_{mod} of the diffracting electrons; hence the diffraction pattern shifts.

Thus we have a new description of interference. Instead of a quantum wave that passes through all the slits of a grating, we have a quantum particle that obeys a nonlocal equation of motion. This equation of motion is exact if the grating is infinite and the incoming beam is an eigenstate of momentum. And even if $V(x + L) - V(x)$ depends on x, e.g. for the case of only two slits, the expectation value of $e^{ip_0 L/\hbar}$ obeys a nonlocal equation of motion. Modular variables provide intuition for the behavior of quantum particles, and they lead us to ask new questions. Here is one: Consider a particle approaching a screen with a single open slit. As it reaches the slit, we open a second slit. By opening the second slit, we change the particle's modular momentum. Don't we act at a distance by changing the particle's modular momentum? The answer is that if we can be sure that the particle passes through the first slit, then its initial modular momentum is completely uncertain and there is no way to detect action at a distance. This is one example of the remarkable way in which quantum mechanics reconciles causality and nonlocality. We will see others.

Problems

5.1 Compute the uncertainty $\Delta p = (\langle p^2 \rangle - \langle p \rangle^2)^{1/2}$ for the state

$$\Psi_\alpha(x, 0) = [\psi(x) + e^{i\alpha}\psi(x + L)]/\sqrt{2}\,,$$

where ψ is normalized and vanishes outside an interval of length $2a < L$, and show that Δp for $\Psi_\alpha(x, 0)$ and for ψ is the same.

5.2 Show that if $V(x + L) = V(x)$ for all x, then the Fourier transform $\int_{-\infty}^{\infty} e^{i(k_x - k'_x)x} V(x)dx$ vanishes unless $\hbar(k_x - k'_x) = nh/L$.

5.3 Let Δx be strictly less than L, i.e. for some x_0 let

$$x_0 - L/2 \le x \le x_0 + L/2 \ .$$

Show that $\Delta p \ge h/L$ by showing that p mod h/L is completely uncertain.

*5.4 For a particle of charge e and mass m, we define a modular velocity component v_{mod} along the direction of $\mathbf{L}$ implicitly as

$$mv_{mod} = (\mathbf{p} - e\mathbf{A}/c) \cdot \frac{\mathbf{L}}{L} \bmod h/L \ ,$$

where $L = |\mathbf{L}|$. Modular velocity is gauge invariant, while modular momentum is not.
(a) Prove the operator equality

$$e^{i(\mathbf{p}-e\mathbf{A}/c)\cdot\mathbf{L}/\hbar} = e^{ie[\Lambda(\mathbf{x})-\Lambda(\mathbf{x}+\mathbf{L})]/\hbar c} e^{i\mathbf{p}\cdot\mathbf{L}/\hbar} \ ,$$

where $A = \nabla\Lambda$. (Hint: prove first that $U^\dagger f(\mathbf{p}\cdot\mathbf{L})U = f(U^\dagger[\mathbf{p}\cdot\mathbf{L}]U)$ for any analytic function $f(\mathbf{p}\cdot\mathbf{L})$ and any unitary operator U.)
(b) Show that there is a gauge in which the Hamiltonian in Sect. 5.1, for electrons interacting with both the grating and the lattice of solenoids, is periodic in x with period L.

5.5 Prove that the electrons and grating in Sect. 5.1 can exchange transverse momentum only in multiples of h/L, i.e. that the interaction between the electrons and the grating conserves transverse modular momentum. (See also Prob. 5.4(b).)

5.6 Consider a wave function $\Psi(x)$ defined as follows: for $nL \le x < (n+1)L$, $\Psi(x) = e^{in\alpha}\psi_0(x - nL)$, where $\psi_0(x)$ is a function defined on the interval $0 \le x < L$. Show that p mod $h/L = \hbar\alpha/L$ for this wave function.

*5.7 Prove that the integral in Eq. (5.6) is independent of α for all times t, assuming that the Hamiltonian of the wave functions is $H = p^2/2m + m\omega^2x^2/2$.

*5.8 Show that two functions $f(x)$ and $g(p)$ can commute only if both are periodic and the product of their periods is a multiple of h.

5.9 Given $[L_z, e^{i\phi}] = \hbar e^{i\phi}$, if we define modular angular momentum $L_{mod} = L_z \bmod 10\hbar$, what is ϕ_{mod} such that $[L_{mod}, \phi_{mod}] = 0$?

5.10 (a) Define $\chi_e = \cos(p_eL/\hbar)$ and $\chi_{sol} = \cos(p_{sol}L/\hbar)$. Show that points (χ_e, χ_{sol}) obeying $\cos([p_e + p_{sol}]L/\hbar) = k$ (where k is a constant) form an ellipse with major and minor axes $\sqrt{1 \pm k}$ along the lines $\chi_e = \pm\chi_{sol}$. (See also Fig. 6.7.)
(b) Let χ_e and χ_{sol} obey $\cos([p_e + p_{sol} + c]L/\hbar) = k$ (where c is a constant). Show that $\cos([p_e + p_{sol}]L/\hbar) = k'$ and compute k'.

*5.11 Find a complete set of simultaneous eigenfunctions of x_{mod} and p_{mod}.

*5.12 Derive the commutation relation for N_x and N_p, defined in Sect. 5.4:

$$[N_x, N_p] = -\frac{i}{2\pi} + \frac{i\hbar}{L}\sum_m \delta\left(p - \frac{mh}{L}\right) + \frac{iL}{2\pi}\sum_m \delta(x - Lm) \ ,$$

where the sums run over all integers m.

*5.13 In an experiment to test the Aharonov–Bohm effect, a particle moves in the xy-plane in a superposition of two nonoverlapping wave packets Ψ_- and Ψ_+ that pass on either side of a thin solenoid located at $(0, 0)$. The solenoid is perpendicular to the xy-axis and contains magnetic flux Φ. On the y-axis there is a screen, and Ψ_- and Ψ_+ pass through holes located at $(0, -L/2)$ and $(0, L/2)$, respectively. Show that the distribution of the velocity component v_y changes discontinuously [3] as the wave packets cross the y-axis.
(a) First calculate the change in $\langle e^{im v_y L/\hbar}\rangle$ with the gauge choice $A_x = -\Phi\delta(x)\Theta(y)$, $A_y = 0$, where $\Theta(y)$ is the Heaviside function $\Theta(y) = 1/2 + y/2|y|$ and $d\Theta/dy = \delta(y)$.
(b) Now repeat the calculation with the gauge choice $A_x = -y\Phi/2\pi(x^2 + y^2)$, $A_y = x\Phi/2\pi(x^2 + y^2)$. (See also Prob. 5.4.)
(c) According to Sects. 4.4 and 5.2, the effect of the flux appears only when the two wave packets recombine; according to (a) and (b), however, the flux affects a measurable observable (modular velocity) in the instant that the line between the wave packets crosses the solenoid. Resolve this paradox!

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[2] H. Weyl, *The Theory of Groups and Quantum Mechanics*, trans. H. P. Robertson (New York: Dover) 1931, Sect. 14 (pp. 272–76); see also J. Zak, *Phys. Rev. Lett.* **19** (1967) 1385.

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6 Nonlocality and Causality

Newton suspected the apparent nonlocality of gravity. "That gravity should be innate, inherent and essential to matter so that one body may act upon another at a distance through a vacuum without the mediation of anything else, by and through which their action or force may be conveyed from one to another, is to me so great an absurdity that I believe no man who has in philosophical matters any competent faculty of thinking can ever fall into it" [1]. More than two centuries years later, Einstein confirmed Newton's suspicions: gravitational interactions are local. Einstein's theory of gravity is free of the "absurdity" of action at a distance.

But quantum mechanics is nonlocal, in at least two ways. There are nonlocal quantum correlations. And there are nonlocal quantum equations of motion: modular momentum, for example, obeys a nonlocal equation of motion. (See Sects. 3.4 and 5.5.) Still, nonlocal quantum correlations do not manifest action at a distance; they manifest "passion at a distance" (in Shimony's apt phrase [2]): there is no cause and no effect in quantum correlations. (See Prob. 3.11.) What about nonlocal equations of motion? Quantum equations of motion may be nonlocal, but quantum uncertainty hides action at a distance. In the example of Sect. 5.5, a particle is about to pass through a screen with a single slit. By opening a second slit some distance away, we change how the modular momentum of the particle evolves. Now modular momentum is an observable; we measure it to observe quantum interference. But in this example, the initial modular momentum of the particle is completely uncertain, so we never see action at a distance. Quantum nonlocality has a logic of its own. Modular variables expose this logic, leading us into paradox but not into contradiction.

6.1 Causality and a Piston

Figure 6.1 shows a long cylinder with a particle in it. The cylinder has a seal at one end, and a piston, initially at rest, at the other end. There is no friction between the piston and the cylinder, but the piston is much heavier than the particle and hardly moves if the particle hits it. Attached to the outer end of the piston is a box with two open sides. Suppose that a little ball strikes the box on the piston while the particle inside the cylinder is as far as possible (a distance L) from the piston. The ball bounces through the box on the piston in two completely elastic collisions that move the piston inward a distance $|\delta L|$. (See Fig. 6.2.) Does the particle in the cylinder affect the trajectory of the ball?

Let us first answer this question within classical mechanics. The ball bounces once off the piston, and the piston moves inward. If the ball bounces off the piston a second time before the particle reaches the piston, it regains all its energy and momentum and leaves the piston

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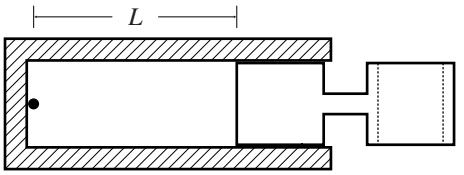

Figure 6.1: A particle in a cylinder with a movable piston.

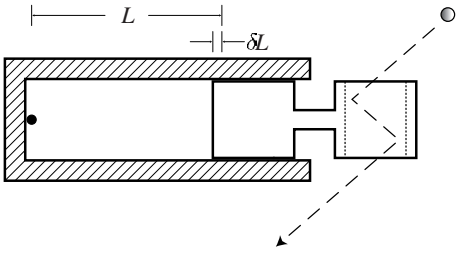

Figure 6.2: A ball bounces off the piston in two elastic collisions, moving the piston inwards a distance δL.

at rest. But if the particle reaches the piston between the two bounces, it carries energy and momentum away from the piston; and when the ball bounces off the piston a second time, it does not regain all the energy and momentum it lost to the piston, and leaves with reduced energy and momentum.

Do we get the same answer in quantum mechanics? Let us take a close look. Again we compare time scales. Suppose the particle is fast and bounces from one end of the cylinder to the other many times between the two collisions of the ball. Its energy increases between the collisions.[1] This increase in energy must come out of the energy of the ball, so the ball loses energy to the piston. But what if the particle is too slow to reach the piston before the second collision? We may assume that the energy of the ball does not change.

Yet the following argument [3] shows that our assumption is incorrect. Let us prepare the particle in an initial state $|\psi(0)\rangle$, at time $t = 0$; the initial state is a wave packet with very large uncertainty Δx in position and very small uncertainty Δp in momentum. Suppose Δx is very large compared with $|\delta L|$ (but small compared with L) and $|\psi(0)\rangle$ is localized near the sealed end of the cylinder, as far as possible from the piston. In the absence of the ball, the particle traverses the cylinder back and forth in a time $T = 2L/|\langle v\rangle|$ where $\langle v\rangle$ is the expectation value of the velocity of the particle. In time T, the wave packet returns to its initial position and momentum, keeping its shape. (See Prob. 6.1.) It spreads out, as Eq. (5.4) shows, but spreads out very little if v is large. We can neglect the spreading if

$$\left(\frac{\Delta p}{p}\right)^2 \ll \left(\frac{\Delta x}{2L}\right)^2 . \tag{6.1}$$

(See Prob. 6.2.)

Now consider the following matrix element,

$$\langle\psi(0)|e^{-iHT/\hbar}|\psi(0)\rangle , \tag{6.2}$$

[1] The increase in energy follows from the quantum adiabatic theorem. (See Chap. 12.) The particle is in a superposition of states $|n\rangle$ with energies $E_n = n^2\pi^2\hbar^2/2mL^2$, where m is the mass of the particle. As L decreases, the energies E_n increase. If the decrease in L is adiabatic, the coefficients of the states $|n\rangle$ remain the same. Hence the expectation value of the energy increases.

with H the Hamiltonian of the particle. On the one hand, Eq. (6.2) is the expectation value of $e^{-iHT/\hbar}$ at time $t = 0$. (Although $e^{-iHT/\hbar}$ is not an observable, its real and imaginary parts are observables.) On the other hand, Eq. (6.2) contains the state of the particle at time $t = T$,

$$|\psi(T)\rangle = e^{-iHT/\hbar}|\psi(0)\rangle \ ,$$

so here the present ($t = 0$) contains the future ($t = T$).

Let us evaluate Eq. (6.2) in two cases. In the first case, the ball does not strike the piston any time before $t = T$. In the approximation that the wave packet does not spread, we can immediately compute Eq. (6.2): $e^{-iHT/\hbar}$ evolves the state $|\psi(0)\rangle$ forward a time T, and $|\psi(0)\rangle$ and $|\psi(T)\rangle$ are the same up to an overall phase. By adjusting L, we can set this phase to 1. (See Prob. 6.3.) Thus

$$|\psi(T)\rangle = |\psi(0)\rangle \tag{6.3}$$

and the expectation value is 1. In the second case, the ball strikes the piston at time $t = 0$, moving it inwards, and again at some time $t \ll T/2$. These two bounces hardly affect $|\psi(0)\rangle$, which vanishes at the piston. However, they change the potential term in the Hamiltonian H. The operator $e^{-iHT/\hbar}$ still evolves the state $|\psi(0)\rangle$ forward a time T, but now $|\psi(T)\rangle$ is displaced a distance $2|\delta L|$ from $|\psi(0)\rangle$, because the cylinder has become shorter inside. We can express this displacement as the action of the translation operator $e^{i2p|\delta L|/\hbar}$; then we obtain, instead of Eq. (6.3),

$$|\psi(T)\rangle = e^{i2p|\delta L|/\hbar}|\psi(0)\rangle \ .$$

For $\Delta x \gg |\delta L|$ the overlap $|\langle\psi(T)|\psi(0)\rangle|$ remains close to 1; but the phase of $|\psi(T)\rangle$ has changed. The change in phase is approximately $e^{i2\langle p\rangle|\delta L|/\hbar}$, where $\langle p\rangle$ is the expectation value $\langle\psi(0)|p|\psi(0)\rangle$. (See Prob. 6.4.) So the ball changes the expectation value of $e^{-iHT/\hbar}$ *already at time* $t = 0$. From the change in this expectation value we infer a small change in the energy of the particle, of roughly $-2\langle p\rangle|\delta L|/T$. It does not matter that we did not compute the change exactly. What matters is that the ball has changed the energy of the particle in the cylinder, even while the particle is far from the piston. By conservation of energy, the energy of the ball has changed, too. Thus a change in the energy of the ball immediately shows that there is a particle in the cylinder, even if the particle is nowhere near the piston!

Now *that* is action at a distance. Suppose that Alice stands at the sealed end of the cylinder and Bob stands near the piston. They agree that, at time $t = 0$, Alice may or may not insert a particle into the cylinder, in the state $|\psi(0)\rangle$. Then Alice sends one bit of information to Bob. By inserting a particle into the cylinder, Alice says Yes to Bob; by leaving the cylinder empty, she says No. Bob measures the energy of the ball just before and just after its two collisions with the piston. A change in the energy means Yes; no change means No. But since the time between the collisions is less than $T/2$, Alice's message reaches Bob before the particle does!

6.2 Quantum Effects Without Classical Analogues

This section presents three quantum effects without classical analogues. Each is likely to surprise us. We see that quantum mechanics implies the effect, and we try to understand the

effect intuitively; yet each new effect surprises us anew. Indeed, we have *not* understood; once we really understand the effects, we see that they are almost the same.

The first effect concerns a particle in a sealed cylinder with a piston (again). Initially, the inner length of the cylinder is L and the particle is in an eigenstate of energy with wavelength $\lambda = 2L/N$, with N large. If we suddenly pull the piston out a distance δL, the wave function of the particle has no time to adjust. What happens to the kinetic energy E of the particle? Specifically, what happens if $\delta L = \lambda/2$, and what happens if δL is slightly less than $\lambda/2$?

If $\delta L = \lambda/2$, the energy is overwhelmingly likely to remain the same. The energy depends on the wavelength, and the wavelength will most likely remain the same – the eigenstate of the new Hamiltonian will resemble the initial eigenstate, but with one additional node. The inner product between these states is $L^{1/2}(L + \lambda/2)^{-1/2}$, so the probability that the energy stays the same is

$$\frac{L}{L + \lambda/2} = \frac{N}{N+1} \ ,$$

which is nearly 1. Now suppose $\delta L = (\lambda/2)(1-\epsilon)$, where ϵ is small and positive. Once again, the most probable new eigenstate is the one with just one more node; however, the wavelength λ' in the new eigenstate is slightly shorter than λ. Since

$$(N+1)(\lambda'/2) = N(\lambda/2) + (\lambda/2)(1-\epsilon) \ ,$$

we have

$$\lambda' = \lambda \left(1 - \frac{\epsilon}{N+1}\right) \ .$$

The kinetic energy E' of the new eigenstate is therefore

$$E' \approx E + \frac{\pi^2 \hbar^2 \epsilon N}{mL^2} \ ,$$

and the energy of the particle in the cylinder has increased! There are other final states with kinetic energy less than E; indeed, the expectation value of the energy does not change if we suddenly pull out the piston. But the probability that the particle has final energy $E' > E$ is substantial. (See Prob. 6.6.) This effect has no classical analogue; classically, the energy of the particle can only decrease if we pull out the piston. (Since $E' - E$ is proportional to $\hbar^2$, the increase in energy vanishes in the classical limit.)

The second effect involves noninteracting one-dimensional particles in a momentum eigenstate. The particles are free except for a very brief time T when they are subject to the potential $V(x)$ shown in Fig. 6.3. The corresponding force field is periodic in the position x: it equals a constant F for $nL \le x \le nL + a$, with n any integer, and vanishes elsewhere. According to classical mechanics, the force affects only the fraction a/L of particles that happen to be in the regions where it is nonzero; each of these particles gets an impulse FT in the direction of increasing x. If a/L is small and there are not many particles, the force may not affect any of them.

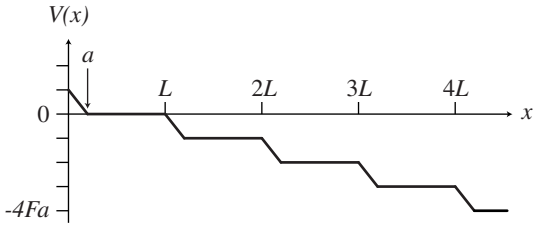

Figure 6.3: The potential $V(x)$ corresponds to a force periodic in space.

According to quantum mechanics, something quite different happens. Let us compute the effect of the potential on an initial state $\Psi_{in}(x)$ of zero momentum, $\Psi_{in}(x) = 1/\sqrt{2\pi}$. The final state $\Psi_{fin}(x)$ after a time T is

$$\Psi_{fin}(x) = e^{-iHT/\hbar}\Psi_{in}(x) \ ,$$

where H is the sum of the kinetic term $p^2/2m$ and the potential term $V(x)$. These terms do not commute. However, if we fix the product $\alpha = aFT$ and take the limit $T \to 0$, we can neglect the kinetic term. (See Prob. 6.8.) The final state $\Psi_{fin}(x)$ is

$$\Psi_{fin}(x) = e^{-iV(x)T/\hbar}\Psi_{in}(x) = \frac{1}{\sqrt{2\pi}}e^{-iV(x)T/\hbar} \ ,$$

where $V(x)T$ does not depend on T since $\alpha = aFT$ does not. To see what $\Psi_{fin}(x)$ represents, we compute its Fourier transform $\tilde{\Psi}_{fin}(p)$. For simplicity, we take a/L to be small and compute the Fourier transform in the limit $a/L \to 0$. (See also Prob. 6.9.) We obtain

$$\begin{aligned}\tilde{\Psi}_{fin}(p) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar}\Psi_{fin}(x) \\ &= \frac{\hbar}{2\pi ip}\left[e^{-ipL/\hbar} - 1\right]\sum_{n=-\infty}^{\infty} e^{in(\alpha - pL)/\hbar} \ . \end{aligned} \tag{6.4}$$

We apply the following identity:

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inz} = \sum_{n=-\infty}^{\infty}\delta(z - 2\pi n) \ . \tag{6.5}$$

(The proof of this identity is Prob. 6.10(a).) Thus

$$\tilde{\Psi}_{fin}(p) = \frac{\hbar}{ip}\left[1 - e^{-ipL/\hbar}\right]\sum_{n=-\infty}^{\infty}\delta(pL/\hbar - \alpha/\hbar - 2\pi n) \ . \tag{6.6}$$

For $\alpha = 0$, $\tilde{\Psi}_{fin}(p)$ reduces to $\delta(p/\hbar)$, the initial state. But for α nonzero, there is a shift in the momentum. The possible values of p are $p = \alpha/L - nh/L$, where n is any integer. These values are not equally probable. The probability of p in the final state is proportional to $\sin^2(pL/2\hbar)/p^2$. Now let us take $\alpha = (2\pi - \epsilon)\hbar$, where ϵ is small and positive. Fig. 6.4 shows the probability distribution for p in the final state. The most likely value of p is $p = -\epsilon\hbar/L$.

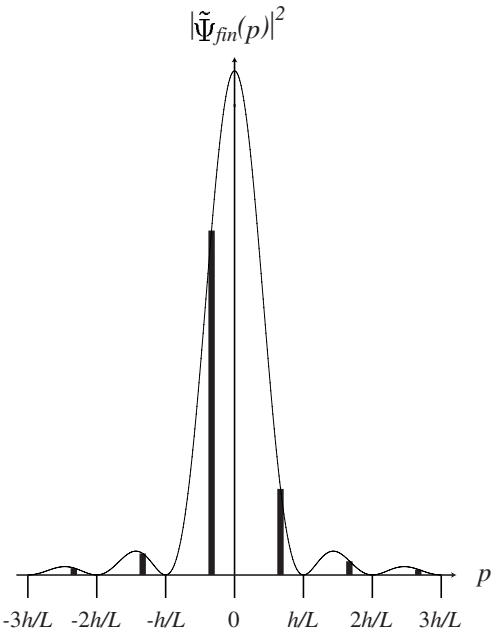

Figure 6.4: The relative probability of allowed values of p in the final state of Eq. (6.6).

Thus, a force to the *right* at points $x = nL$ most likely moves quantum particles to the *left*! This is a purely quantum effect, vanishing in the limit $\hbar \to 0$.

The third effect involves a spin-1/2 particle of magnetic moment μ in a magnetic field. The Hamiltonian for the particle is

$$H = \frac{\mu\hbar}{2}\left[\sigma_z B_z + \sigma_z B_z'(t) + \sigma_x B_x(t)\right] \ ,$$

where B_z (pointing in the z-direction) is constant while $B_z'(t)$ (also pointing in the z-direction) and $B_x(t)$ (pointing in the x-direction) are periodic in time. If $B_z'(t)$ and $B_x(t)$ are zero, the particle simply precesses in the constant magnetic field with angular frequency $\omega = \mu B_z$, and $S_z = \hbar\sigma_z/2$ is a constant of the motion. Suppose $B_z'(t) = 0$ but $B_x(t)$ pulses with period T. (See Fig. 6.5(a).) Will the spin flip? It will flip if T is any multiple of the precession time. The particle makes a full precession in a time $2\pi/\omega = 2\pi/\mu B_z$. If T is a multiple of this time, i.e. if

$$T = 2n\pi/\mu B_z \tag{6.7}$$

for integer n, then the effect of $B_x(t)$ is cumulative and the spin flips. Otherwise, the effect of $B_x(t)$ averages to zero. Suppose that both $B_z'(t)$ and $B_x(t)$ pulse with period T, but never at the same time, as in Fig. 6.5(b). Even if T does not satisfy Eq. (6.7), we can still flip the spin by choosing $B_z'(t)$ appropriately. This effect has a simple classical explanation. The field $B_z'(t)$ adjusts the particle's precession so that the particle makes one or more full precessions

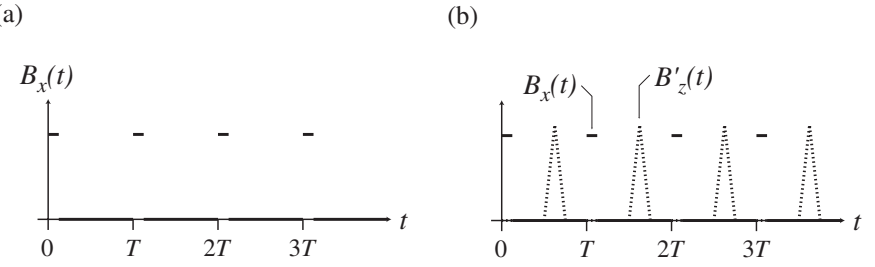

Figure 6.5: (a) $B_x(t)$ as a function of time, with $B'_z(t) = 0$. (b) $B_x(t)$ and $B'_z(t)$ as functions of time.

between every $B_x(t)$ pulse, even if T does not satisfy Eq. (6.7); then the effect of $B_x(t)$ is still cumulative. The same explanation applies to the quantum particle. So where is the quantum effect without a classical analogue?

The answer is that the effect has a classical analogue if we look at precession, but not if we look at energy. The change of energy in a flip between $|\uparrow\rangle$ and $|\downarrow\rangle$ (eigenstates of σ_z) is $\pm\mu\hbar B_z$. What accounts for this energy change? The $B'_z(t)$ field briefly changes the particle's energy, but the change averages to zero. The proof is that $[\sigma_z B_z + \sigma_x B_x(t), \sigma_z B'_z(t)]$ vanishes (since $B_x(t)B'_z(t)$ vanishes) so the average values of $\sigma_z B_z + \sigma_x B_x(t)$ and $\sigma_z B'_z(t)$ are constant in time. Thus, the particle can exchange energy only with the $B_x(t)$ field.

But if $B'_z(t)$ vanishes, the particle and the $B_x(t)$ field (i.e. the terms $\mu\hbar\sigma_z B_z/2$ and $\mu\hbar\sigma_x B_x(t)/2$ in H) can exchange energy only in multiples of h/T. This fact follows from a perturbative calculation. Suppose we subject the particle to a perturbation $g_0 V \sin\omega t$ lasting a time T/g_0, where g_0 is a constant, the operator V is time independent, and $\omega \approx \mu B_z$. The probability of a spin flip is approximately

$$\left| \langle\uparrow |V| \downarrow\rangle g_0 \frac{\sin(\mu B_z T/g_0 - \omega T/g_0)}{\hbar(\mu B_z - \omega)} \right|^2 . \tag{6.8}$$

(See Prob. 6.11.) Now we consider the limit $g_0 \to 0$; we find that the spin flips only if $\mu B_z = \omega$. The limit $g_0 \to 0$ is pertinent because the perturbation can be arbitrarily long and weak; as long as $B_x(t)$ has a cumulative effect on the spin, the spin will eventually flip. The Fourier components of $B_x(t)$ have angular frequencies that are multiples of $2\pi/T$, so by applying Eq. (6.8) (or Eq. (2.3)) we find that $B_x(t)$ supplies energy in quanta $h\nu = h/T$, i.e. the spin flips only if $\mu\hbar B_z = nh/T$. So we are back to Eq. (6.7).

To summarize:

i) If the $B'_z(t)$ field is zero, the particle and the $B_x(t)$ field exchange energy only in quanta h/T. The spin does not flip unless the spin energy gap is a multiple of h/T.

ii) If the $B'_z(t)$ field is nonzero, it exchanges no energy with the particle and the $B_x(t)$ field. But the spin flips even if the spin energy gap is not a multiple of h/T.

Apparently, the $B'_z(t)$ field modifies the exchange of energy between the particle and the $B_x(t)$ field, while making no net contribution to their energy! This effect is nonlocal in *time* since $B'_z(t)$ vanishes whenever the particle and the $B_x(t)$ field interact.

6.3 Modular Energy

The last two sections describe quantum effects without classical analogues. The Aharonov–Bohm effect (Sect. 4.4) is another such effect. What do all these effects have in common? They are all nonlocal. Even the first effect in Sect. 6.2 (where pulling back a piston in a cylinder increases the energy of a particle inside) is nonlocal in that local interactions (collisions between the particle and the pulled piston) can only decrease the energy of the particle. What else do the effects have in common?

Let's return to the Aharonov–Bohm effect and consider an electron in a long, narrow box. Figure 6.6(a) shows the initial state of the electron. The wave function is symmetric around the vertical axis in Fig. 6.6(a) and vanishes in a central region. A long, thin solenoid carrying a flux $hc/2e$ – half a flux quantum – crosses the box through the central region in a short time T. It leaves the box before the electron wave function can spread to the central region, hence the electron never touches the solenoid. However, the solenoid induces a relative phase of -1 between the two halves of the wave function on either side of the central region. (The Aharonov–Bohm phase depends only on the relative motion of the solenoid and the electron.) Once the solenoid passes, the state of the electron is antisymmetric in the vertical axis of Fig. 6.6(b). The symmetric and antisymmetric states have different energies. Thus the passage of the solenoid has changed the distribution of energy of the electron. Yet the energy distribution of the solenoid does not change, because the solenoid never encounters the electron.

We can analyze this nonlocal effect as follows. Let ρ_e and ρ_s denote the probability distributions for the energies of the electron and the solenoid, respectively, and ρ_{es} the probability distribution for their combined energy; ρ_{es} is a convolution of ρ_e and ρ_s:

$$\rho_{es}(E) = \int \rho_e(E - E')\rho_s(E')dE' \ . \tag{6.9}$$

If total energy is conserved, $\rho_{es}(E)$ does not change with time. Also, $\rho_s(E')$ does not change. But $\rho_e(E - E')$ changes. From Eq. (6.9) we do not see just how $\rho_e(E - E')$ can change while $\rho_{es}(E)$ and $\rho_s(E')$ do not, but from the Fourier transform of Eq. (6.9) we do. By definition,

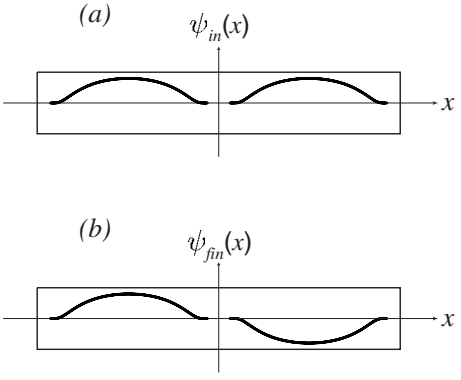

Figure 6.6: (a) Initial state of the electron in the box of Sect. 6.3. (b) Final state of the electron after a solenoid carrying half a flux quantum crosses the box at $x = 0$.

the Fourier transform of $\rho_s(E)$ is

$$\begin{aligned}\tilde{\rho}_s(t) &= (2\pi)^{-1/2}\int e^{iEt/\hbar}\rho_s(E)dE \\ &= (2\pi)^{-1/2}\langle\psi_s|e^{iH_st/\hbar}|\psi_s\rangle\ ,\end{aligned} \tag{6.10}$$

where ψ_s is the initial wave function of the solenoid and H_s is its Hamiltonian. The Fourier transform of Eq. (6.9) is

$$\tilde{\rho}_{es}(t) = \tilde{\rho}_e(t)\tilde{\rho}_s(t)\ .$$

Now ρ_e can change without ρ_s and ρ_{es} changing only if $\tilde{\rho}_e$ changes without $\tilde{\rho}_s$ and $\tilde{\rho}_{es}$ changing. That is, $\tilde{\rho}_s(t)$ and $\tilde{\rho}_{es}(t)$ must vanish over some interval in t. The term $\langle\psi_s|e^{iH_st/\hbar}$ in the second line of Eq. (6.10) represents the wave function of the solenoid after a time t. If $\tilde{\rho}_s(t) = 0$, then after a time t the solenoid has evolved to an orthogonal state. The solenoid does indeed evolve to an orthogonal state: it crosses the box in a time T. Hence if $t \geq T$, then $\tilde{\rho}_s(t)$ vanishes.

We can interpret $\langle\psi_s|e^{iH_st/\hbar}|\psi_s\rangle$ in Eq. (6.10) as the expectation value of $e^{iH_st/\hbar} = e^{i(H_s)_{mod}t/\hbar}$, where we define the *modular energy* $(H_s)_{mod}$ or $(E_s)_{mod}$ of the solenoid to be its energy modulo h/t, in analogy with the definitions of p_{mod} and x_{mod} (Sects. 5.3–4). Equation (6.10) shows that the modular energy of the solenoid is completely uncertain if $t \geq T$. What does it mean for the modular energy to be completely uncertain? Suppose the modular energy of the *electron* changes. Then so does the modular energy of the solenoid: the sum of energies $E_e + E_s$ does not change, hence the modular energies of the solenoid and the electron satisfy

$$\cos(E_e + E_s)t/\hbar = \text{constant}\ . \tag{6.11}$$

Equation (6.11) defines an ellipse. (See Fig. 6.7 and Prob. 5.10.) By measuring the modular energy of the solenoid immediately before and after it crosses the box, we could determine any change in the modular energy of the electron. But since the solenoid never encounters the electron, any measurable change in $(E_s)_{mod}$ would be action at a distance. Can we observe action at a distance? We cannot, because the modular energy of the solenoid is completely uncertain.

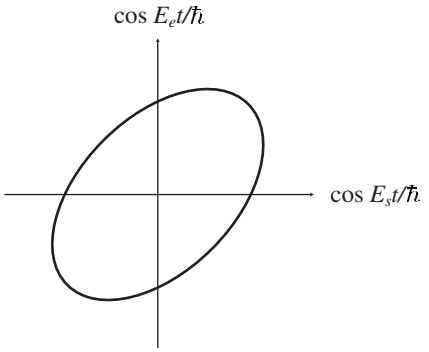

Figure 6.7: Modular energies of an electron and a solenoid according to Eq. (6.11). (See also Prob. 5.10.)

Here is the logic of quantum nonlocality. The electron and the solenoid interact nonlocally, exchanging modular energy. The nonlocal interaction is well defined only *after* the solenoid crosses the box; but after the solenoid has crossed the box, its modular energy is completely uncertain. Indeed E_s mod h/T is completely uncertain at all times, since the solenoid moves to an orthogonal state in a time T. Hence the nonlocal exchange of modular energy has no observable effect on the solenoid.

Modular energy and momentum are the key to the effects in Sects. 6.1–2. In Sect. 6.1, the presence of a particle in the cylinder shows up as a change in the energy of the ball. Specifically, Eq. (6.2) implies that it is the *modular* energy – the energy modulo h/T – of the the ball that changes, if and only if Alice puts a particle in the cylinder. Bob must measure the change in the modular energy of the ball in order to detect the presence of the particle. He must measure the modular energy of the ball before and after its two collisions with the piston, and he must do so within a time less than T to demonstrate action at a distance. It follows that in a time less than T the ball must have bounced twice and evolved to an orthogonal state, i.e. that

$$\langle\Psi(0)|e^{-iH_{ball}T/\hbar}|\Psi(0)\rangle = 0 \ ,$$

where $|\Psi(0)\rangle$ is the state of the ball at the first bounce and H_{ball} is the Hamiltonian of the ball. Thus E_{ball} mod h/T is completely uncertain[2] and Bob cannot detect the particle that Alice may or may not have put in the cylinder. Hence Alice *cannot* send a message to Bob by action at a distance. In Sect. 6.2, the first effect involves an exchange of modular energy $E_{mod} \approx E$ mod $\hbar^2\pi^2 N/mL^2$; increases in energy are due to exchange of modular energy, while decreases in energy are due to local interactions between the particle and the piston. In the second effect, the unexpected motion of particles is due to exchanges of modular momentum, and in the third effect the particle acquires modular energy from $B_z'(t)$. (See Probs. 6.7 and 6.12.)

6.4 Reconciling the Irreconcilable

Quantum theory is implicitly nonlocal. Nonlocality is a wonder, but we may also wonder why quantum nonlocality is implicit rather than explicit. Why do we have to work so hard to demonstrate quantum nonlocality? Why isn't quantum mechanics *more* nonlocal than it is? For example, why can't we exploit quantum nonlocality to send messages? Chapters 3–6 contain many examples of quantum nonlocality. But in each example, we encounter a constraint. What constraints does quantum nonlocality obey?

Let us try to guess a constraint: *quantum nonlocality does not allow action at a distance.* Nonlocal correlations, we know, obey this constraint. (See Prob. 3.11). As for modular variables, they behave nonlocally – by opening one slit, we change the modular momentum of a particle passing through another slit – but in the example of Sect. 5.5, quantum uncertainty prevents us from measuring the change, so we cannot exploit the change in modular momentum to act at a distance. The examples of this chapter expose the same logic: quantum uncertainty hides changes in modular variables that would otherwise manifest action at a distance. This logic hints at a fundamental constraint.

[2] According to the complete uncertainty principle; see Sect. 5.3.

Nevertheless, the constraint we guessed is not right. To see what is wrong, consider the scalar Aharonov–Bohm effect. (See Sect. 4.4.) Electrons interfere along two paths on either side of a capacitor. By charging the capacitor briefly as the electrons pass, we change their interference pattern. We can interpret this effect as the *local* effect of a scalar potential $V(\mathbf{x}, t)$. However, the potential $V(\mathbf{x}, t)$ is not measurable; we can only measure the gradient of $V(\mathbf{x}, t)$ (which vanishes along the two paths). To make the correspondence between theory and measurement as close as possible, we choose the interpretation of Sect. 4.5: the change in the electrons' interference pattern is a *nonlocal* effect of the electric field in the capacitor. This nonlocal effect is action at a distance! The electric field, acting at a distance on the electrons, yields a measurable effect (the change in the electrons' interference pattern). So quantum nonlocality *does* permit action at a distance.

There is a constraint similar to the one we guessed. It is called (relativistic) *causality*. The principle of causality states that there is no way to send a message faster than light. We do not expect nonrelativistic quantum mechanics to obey this principle, because in nonrelativistic quantum mechanics there is no maximum speed. However, nonrelativistic quantum mechanics obeys the principle of causality more than we might expect. Nonlocal quantum correlations obey causality, because they are useless for sending messages. The Aharonov–Bohm effect, too, obeys causality: even if an electric or magnetic field acts at a distance on electrons as they pass, the only measurable effect – the change in the electrons' interference pattern – lies within the future light cone of the field. (See Prob. 6.14.) In each example we find that *quantum nonlocality obeys causality.*

As Sect. 6.3 shows, the paradox of Sect. 6.1 respects causality because the modular energy of the ball that moves the piston, E_{ball} mod h/T, is completely uncertain. But E_{ball} mod h/T may *not* be completely uncertain, and then the paradox respects causality in a different way. Let the ball move along its trajectory in a superposition of two identical wave packets, with a time lag T between the wave packets. Then $\langle\Psi(0)|e^{-iH_{ball}T/\hbar}|\Psi(0)\rangle$ does *not* vanish and E_{ball} mod h/T is measurable. But then the uncertainty about when the ball struck the piston is greater than T and the particle inside the cylinder could have reached the piston before the ball.

It is remarkable that quantum theory reconciles nonlocality and causality. They seem incompatible. Is quantum theory the *only* theory that reconciles causality and nonlocality? A positive answer to this question would transform our understanding of quantum theory, for then we could *deduce* quantum theory from causality and nonlocality. Quantum theory would follow from two axioms:

i) All physical interactions respect causality.

ii) Some physical interactions are nonlocal.

Quantum theory would then be as logically simple as the special theory of relativity, which follows from two axioms:

i′) The laws of physics are the same in all inertial reference frames.

ii′) The speed of light in vacuum is c.

Each of the axioms i′) and ii′) has a clear physical meaning. Axiom i′) states a fundamental invariance; axiom ii′) defines a physical constant. These axioms are the starting point of the special theory of relativity. If we chose another starting point, such as the dependence of length and time on motion, the theory would look quite bizarre and meaningless. Why should objects contract in the direction of their motion? Only after discovering the axioms i′) and ii′)

would we understand the theory. Perhaps the problem of understanding quantum mechanics is analogous. Quantum mechanics looks bizarre, because our axioms – statements about Hilbert space and Hermitian operators – lack clear physical meaning. But axioms i) and ii) have clear physical meaning.

Do axioms i) and ii) imply quantum theory? We return to this question in the last chapter of the book. Here, we pose the question and suggest how quantum mechanics might plausibly follow from these axioms. From i) and ii) we can deduce uncertainty: physical variables that interact nonlocally cannot all have definite values at all times. Otherwise there would be observable violations of causality. For example, any quantity that is exchanged nonlocally must be unmeasurable. Causality requires modular momentum and modular energy to be completely uncertain at times. Alice, by inserting a particle into the cylinder of Sect. 6.1, changes the modular energy of the piston nonlocally – action at a distance. But the modular energy of the piston is completely uncertain. We have action at a distance with no observable results.

Yet action at a distance can have observable results. The "interaction-free" measurement of Elitzur and Vaidman [4] is an example. To dramatize the interaction-free measurement, let us imagine a "bomb" or "mine" so sensitive that any interaction makes it explode. Is there any way we could detect such a bomb without exploding it? To detect a bomb in a region $\mathcal{S}$, we build a Mach-Zehnder interferometer with one of its arms crossing the region $\mathcal{S}$. Figure 6.8(a) shows the interferometer with no bomb in the region $\mathcal{S}$. A half-silvered mirror splits the incident photon beam into two equal parts. The parts recombine at another half-silvered mirror. By adjusting the length of each arm (and taking into account the $\pi/2$ phase difference between reflection and transmission), we can make the parts of the beam interfere constructively in one direction and destructively in the other. Then all the photons leave the interferometer in the same direction. If, however, a detector records which path the photon actually takes through the interferometer, interference disappears (according to the complementarity principle) and the photon may leave the interferometer in either direction. If there is a bomb in the region $\mathcal{S}$ and it does not explode, it records the fact that the photon did not pass through $\mathcal{S}$. The photon went through the other arm of the interferometer. Then interference disappears and the photon may come out either way. (See Fig. 6.8(b).) So if we see a single photon leave the interferometer in the direction of destructive interference, we have detected the bomb without exploding it.

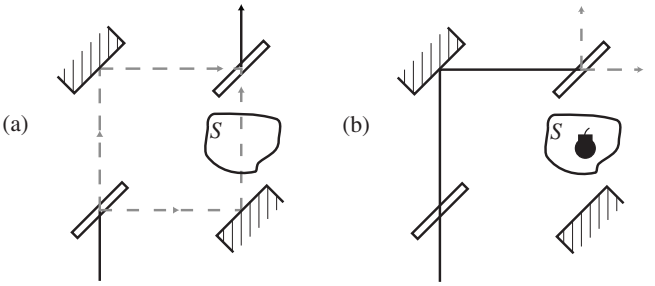

Figure 6.8: (a) Mach-Zehnder interferometer with one arm crossing a region $\mathcal{S}$. (b) The interferometer with a "bomb" in region $\mathcal{S}$.

This baffling quantum effect nicely illustrates how modular variables work. The two possible final states of the photon – the possible directions in which it may leave the interferometer – correspond to different values of a modular variable. Which modular variable? Consider a plane passing through the two half-silvered mirrors; let an operator P reflect the photon wave function through this plane. (See Prob. 6.15.) The expectation value of P is $\cos\alpha$, where α is the relative phase between the two parts of the photon wave function in the two arms of the interferometer. The operator corresponding to α is a modular variable: it is nonlocal and has the topology of an angle. The possible final states of the photon correspond to different distributions of α. When both arms of the interferometer are clear, the value of α is defined and the photon leaves the interferometer in the direction of Fig. 6.8(a). When one arm is obstructed, α is completely uncertain and the photon leaves the interferometer in either direction. The "bomb" acts at a distance on the photon by changing the expectation value of P.

Problems

6.1 Show that a wave packet reflecting from an infinite potential barrier acquires a phase factor -1 but keeps its shape.

6.2 Let $\Psi(x,t)$ denote the gaussian wave packet of Eq. (5.4).
(a) Show that

$$\left|\int_{-\infty}^{\infty} dx \Psi^*(x,t)\Psi(x-\hbar kt/m,0)\right|^2 = \left(1+\frac{\hbar^2 t^2}{4m^2a^4}\right)^{-1/2} .$$

(b) Derive Eq. (6.1) from the answer to Prob. 6.2(a).

6.3 In Eq. (6.2), $|\psi(0)\rangle$ is a superposition of eigenstates of H with eigenvalues $E_n = \hbar^2 n^2\pi^2/2mL^2$. Assume that in this superposition $n \approx N$, i.e. $(n-N)^2/N \ll 1$, and let $T = 2mL^2/\hbar\pi N$. Compute $\langle\psi(0)|e^{-iHT/\hbar}|\psi(0)\rangle$ and show that it approaches $e^{-iN\pi}$ for large N.

6.4 Let $\psi_g(x)$ denote the wave packet of Eq. (5.3) and evaluate

$$\int_{-\infty}^{\infty} dx \psi_g^*(x) e^{i2p|\delta L|/\hbar}\psi_g(x)$$

for $|\delta L|/a \ll 1$.

6.5 Consider a particle moving in a one-dimensional box of length L, in a wave packet with expectation value $\langle v_x\rangle$ for the velocity. At $t=0$ we open a small hole at one end, and the wave packet evolves into a train of little wave packets of diminishing size. Estimate the particle's modular energy.

6.6 (a) For the first effect in Sect. 6.2, let the initial energy E of the particle in the cylinder be $E = h^2/2m\lambda^2$ and let the initial (inner) length of the cylinder be 4.5λ. Suppose that the length increases suddenly to 4.9λ. Show that the sudden increase has an 80% chance of increasing the energy by 4%.
(b) Show that the expectation value of the energy does not change.

6.7 Compute the change in modular energy of the particle in the first effect of Sect. 6.2.

*6.8 Let H be the Hamiltonian

$$H = \frac{p^2}{2m} + \delta(t)V(x) \ ,$$

with $\delta(t) = 1/T$ for $0 \le t \le T$ and $\delta(t) = 0$ otherwise. Assume that $dV(x)/dx$ and $d^2V(x)/dx^2$ are finite for all x. Show that

$$|\psi(t)\rangle = e^{-iV(x)t/T\hbar}|\psi(0)\rangle$$

solves the Schrödinger equation $i\hbar\partial|\psi(t)\rangle/\partial t = H|\psi(t)\rangle$ for $0 \le t \le T$ in the limit $T \to 0$. (We apply this result to the potential $V(x)$ in Fig. 6.3 by smoothing $V(x)$ at the points $\ldots, -L, -L+a, 0, a, L, L+a, \ldots$.)

*6.9 Show that the expectation value of p in the state $|\tilde{\Psi}_{fin}\rangle$, according to Eq. (6.6), is

$$\langle\tilde{\Psi}_{fin}|p|\tilde{\Psi}_{fin}\rangle = \frac{\hbar}{L}\sin(\alpha/\hbar) \ ,$$

which can be negative. But according to Ehrenfest's theorem, it equals the product of the time T and the expectation value of the force $-dV/dx$, which is positive! (See Prob. 4.4.) Show that this contradiction is due to the unphysical limit $a/L \to 0$ taken in Eq. (6.4).

*6.10 (a) Prove Eq. (6.5) by inserting the left-hand side into an integral, restricting the sum to $|n| \le M$, performing the integral by contour integration, and taking the limit $M \to \infty$. (b) Show that for $\alpha = 0$, $\tilde{\Psi}_{fin}(p)$ in Eq. (6.6) reduces to $\delta(p/\hbar)$.

6.11 Let $|E\rangle$ and $|E'\rangle$ be nondegenerate eigenstates of a Hamiltonian H, with eigenvalues E and E', respectively. At time $t = 0$, the state of the system is $|E\rangle$. Consider a perturbation $H \to H + g_0V\sin\omega t$ for $0 \le t \le \tau$, where g_0 is a constant and V is time independent. Solve the time-dependent Schrödinger equation to first order in g_0. Show that the probability for the system to be in the state $|E'\rangle$ at time $t = \tau$ is

$$g_0^2|\langle E|V|E'\rangle|^2\left|\frac{e^{i(E'-E)\tau/\hbar+i\omega\tau}-1}{E'-E+\hbar\omega} - \frac{e^{i(E'-E)\tau/\hbar-i\omega\tau}-1}{E'-E-\hbar\omega}\right|^2$$

to this order [5].

*6.12 For the third effect in Sect. 6.2, define an appropriate modular energy and show that the field $B'_z(t)$ changes the modular energy of the spin-1/2 particle.

*6.13 Consider a spin-1/2 particle restricted to the x-axis. It oscillates harmonically, with a large amplitude A, keeping its shape. Initially, $S_z = -\hbar/2$. A magnetic field $B(t)$, parallel to the z-axis, turns on briefly every time the particle reaches the point $x = -A$; thus the energy of the particle changes by $\mu\hbar B(t)/2$ but no force acts upon it. In a neighborhood of $x = A$, a passing pellet strikes the particle and flips its spin. Calculate the change in modular energy E_{mod} of the pellet. Show that the change in E_{mod}, which depends nonlocally on $B(t)$, is consistent with nonrelativistic causality.

6.14 The scalar Aharonov–Bohm effect involves a transient potential difference, hence a transient electric field, between two interfering paths. Along the paths, electric and magnetic fields vanish. Show that the Aharonov–Bohm phase is measurable only within the forward light cone of the transient electric field, hence the effect does not allow superluminal signalling.

*6.15 Section 6.4 discusses the "bomb" detector of Elitzur and Vaidman and defines an operator P as reflection through a plane passing through both half-silvered mirrors. Define the x-axis to be perpendicular to the plane, so that P is the one-dimensional parity operator for x. Find a unitary representation of P such that $PxP^\dagger = -x$ and $PpP^\dagger = -p$, where x and p are operators. (Hint: harmonic oscillator eigenstates are eigenstates of P.)

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[5] See, for example, L. I. Schiff, *Quantum Mechanics*, Third Edition (New York: McGraw-Hill), 1968, pp. 282–84.

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7 Quantum Measurements

In physics, we expect a theory to correspond with experiments in two ways. One correspondence is obvious: the theory should correctly predict the results of experiments. But there is another, more subtle correspondence. The theory should predict not *less* and not *more* than the results of possible experiments. That is, the theory should predict anything that experiments can ever test, but it should not predict anything that experiments can never test. Does a theory describe too little? Too much? These simple questions are among the most powerful we can ask in criticizing a theory. They have led to great progress in physics. For example, Newton, in his mechanics, made assertions about "absolute space" and "absolute time". Mach questioned these and other assertions of Newton's mechanics. How do we define motion if there is nothing else in the universe? We must define it relative to some reference frame such as the stars in the sky, or whatever else we take to be motionless. In empty space we could never define motion, so absolute space and time have no meaning. Mach's critique helped Einstein arrive at the special and general theories of relativity. Newton's mechanics describes *too much*. A theory that describes *too little* is thermodynamics; it accounts only for the bulk properties of gases. (To account for the microscopic properties of molecules in gases, we need statistical mechanics.) The special theory of relativity comes the closest to the ideal correspondence between theory and experiment.

Thus, we can deepen our understanding of quantum mechanics by asking how well the physical statements of the theory apply to experiments. The discussions of Bohr and Einstein on quantum theory, from 1927 onwards, were devoted to this question. Einstein kept trying to show that experiments can measure more than quantum mechanics allows, i.e. that quantum mechanics describes *too little*. The fact that he did not succeed does not prove that he was wrong. For example, the two thought experiments of Chap. 4 present a new challenge to quantum mechanics, unknown to Einstein. We also encounter (in Chap. 11) the claim that quantum mechanics describes *too much*, that there is no experiment that can distinguish among certain quantum states. (Too much is better than too little.)

We begin this chapter with a paradox concerning a measurement of the velocity of a quantum particle. This paradox apparently implies that the Heisenberg uncertainty relations do not completely account for what we can and cannot measure in quantum mechanics. We resolve this paradox by treating the device for measuring velocity *as a quantum system*. But *can* we treat a measuring device as a quantum system? Bohr treated measurements via an *ad hoc* division of the world into quantum and classical parts, with measuring devices residing in the classical part. This description served him in his discussions with Einstein, allowing him to resolve the clock-in-the-box paradox and others. (See Sect. 2.4.) However, the division of the world into classical and quantum parts is not well defined. Bohr never prescribed how to divide

Quantum Paradoxes: Quantum Theory for the Perplexed. Y. Aharonov and D. Rohrlich

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the world into classical and quantum parts; the division arises in practice, not in principle.[1] If so, why not let the measuring device reside in the quantum part of the world? Such a treatment does not resolve the question of where in the world is the dividing line (a question Chap. 9 addresses) but it has an outstanding advantage: if we treat the measuring device as a quantum system, we automatically endow it with all the limitations and possibilities of a quantum system. We need not, as Bohr and Einstein did, face each new question of measurement by asking what limitations quantum mechanics imposes on a classical measuring device. Moreover, a quantum measuring device may respond in ways we never anticipated. Treating the measuring device as a quantum system allows us to consider (in Chaps. 14–17) quantum measurements that are unlike any classical measurements.

7.1 The Velocity Paradox

Quantum systems present much more difficult and interesting problems of measurement than classical systems. We can make any measurement on a classical system and not disturb it, in principle. But as Chap. 2 relates, some measurements on a quantum system *must* disturb it. This fact led Heisenberg to the uncertainty relations. (See Sect. 2.3.)

What does measuring a quantum system disturb? According to the uncertainty relations, the measured observable need not change. What changes is an observable conjugate to the measured observable. When we measure the momentum of a particle, we assume that its position, but not its momentum, changes. Since the momentum of a free, massive particle is the product of its velocity and its mass, we expect that we can measure its velocity, too, without changing it. We can measure its momentum as quickly as we like (in nonrelativistic quantum mechanics[2]) so we expect that we can measure its velocity as quickly as we like, too.

However, let us consider an experiment on a free particle moving in one dimension. At time $t = 0$ we know the position $x(0)$ of the particle to an accuracy $\Delta x(0)$, from a prior measurement. We also know its velocity $v(0)$ to an accuracy $\Delta v(0) = \hbar/2m\Delta x(0)$, consistent with the uncertainty relations. At $t = 0$ we begin a more accurate measurement of the particle's velocity. This measurement lasts a time T. At time T we know the particle's velocity $v(T)$ with an accuracy $\Delta v(T) \ll \Delta v(0)$. (Fig. 7.1 is a graph of $v(t)$ and $\Delta v(t)$ versus time.) We assume that measuring the velocity does not change it, and that T can be as short as we like.

The position and velocity of the particle are related by the Heisenberg equation of motion, Eq. (5.17):

$$v(t) = \frac{dx(t)}{dt} = \frac{i}{\hbar}[H, x] \; . \tag{7.1}$$

We can integrate Eq. (7.1) to get

$$x(T) - x(0) = \int_0^T v(t)dt \; . \tag{7.2}$$

[1] Such phenomena as the Josephson junction, superconductivity, and magnetoresistance in metal loops show that quantum effects are not restricted to microscopic systems. See Sects. 4.6–7.

[2] See Chap. 14 in regard to relativistic quantum mechanics.

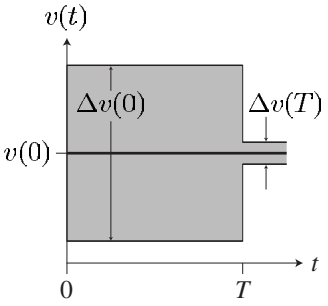

Figure 7.1: Graph of $v(t)$ and $\Delta v(t)$ in the thought experiment of Sect. 7.1.

Note that Eq. (7.2) is an operator equation. During the measurement the particle is not free, so we cannot replace $v(t)$ by p/m. However, if measuring the velocity does not change it, we can replace $v(t)$ by $v(0)$ to get

$$x(T) = x(0) + v(0)T \ .$$

We can estimate the uncertainty in $x(T)$ from the uncertainties in $x(0)$ and $v(0)$:

$$[\Delta x(T)]^2 = [\Delta x(0)]^2 + [\Delta v(0)]^2 T^2 \ . \tag{7.3}$$

For T arbitrarily small, the uncertainty in x hardly increases as a result of the measurement; yet the measurement decreases the uncertainty in the velocity. So once the measurement is complete and the particle moves freely again, the product of uncertainties $\Delta x(T) \cdot \Delta v(T)$ violates the Heisenberg uncertainty relations. Either the Heisenberg uncertainty relations are wrong, or we were wrong in assuming that we can measure velocity in an arbitrarily short time without changing it. Either way, we conclude that the Heisenberg uncertainty relations do not fully account for what we can and cannot measure in quantum mechanics.

What about the time-energy uncertainty relation? Doesn't it apply to the velocity paradox? According to one interpretation of the time-energy uncertainty relation, a measurement of energy to an accuracy ΔE takes a time at least $T \geq \hbar/\Delta E$. Before the velocity measurement began, the particle was free, hence a measurement of its velocity also measured its kinetic energy. If so, we cannot assume that T is arbitrarily short or that $\Delta v(T) \ll \Delta v(0)$, and there is no paradox.

However, we must be careful how we interpret the time-energy uncertainty relation. (See Chap. 8.) Energy is a Hermitian operator, but the time t in Schrödinger's equation is a parameter, not an operator. There is no Hermitian operator T such that $[T, H] = i\hbar$ (the analogue of $[x, p] = i\hbar$). The proof is that $[T, H] = i\hbar$ implies that H has no lower bound; but $H = p^2/2m$ cannot be negative. (See Prob. 7.1.) We can derive a time-energy uncertainty relation from the position-momentum uncertainty relation, as in Prob. 2.8, but we cannot then use it to prove the latter relation.

7.2 A Quantum Measurement Paradigm

As a step towards treating quantum measurements, let us review a standard experiment for measuring a quantum observable – spin. We can extract, from this experiment, a paradigm for all quantum measurements.

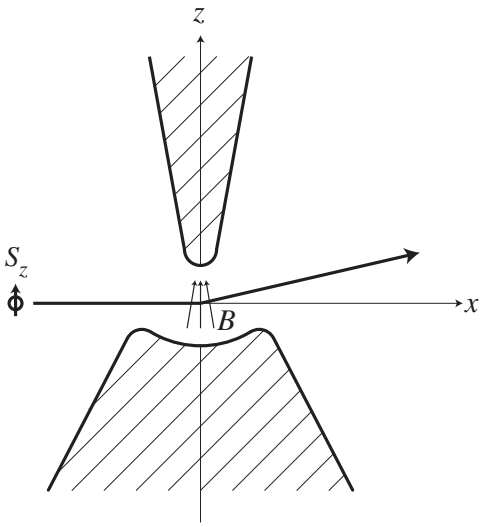

Figure 7.2: An atom passes between two large magnets in a Stern-Gerlach apparatus to measure the spin component S_z of the atom.

A standard experimental setup to measure spin is the Stern-Gerlach apparatus. Figure 7.2 depicts this setup schematically. An atom with z-component of spin S_z enters from the left, along the x-axis. It encounters an inhomogeneous magnetic field $\mathbf{B}$ near the origin. The Hamiltonian for the interaction of the spin with the field is [1]

$$H_{int} = -\mu \mathbf{S} \cdot \mathbf{B} \ ;$$

it is the potential energy of a magnetic dipole of moment $\mu\mathbf{S}$ in a magnetic field $\mathbf{B}$. As the atom passes through the field, its momentum changes in accord with the Heisenberg equation of motion:

$$\frac{d\mathbf{p}}{dt} = \frac{i}{\hbar}[H_{int}, \mathbf{p}] = \mu \nabla (\mathbf{S} \cdot \mathbf{B}) \ .$$

Assume that $\mathbf{B}$ is parallel to the z-axis, with z-component $B_z = Bz$, for simplicity.[3] Then if the atom crosses the magnetic field in a time T, it acquires transverse momentum $\mu(\partial B_z/\partial z)S_z T$, proportional to the spin component S_z. A beam of atoms entering the Stern-Gerlach apparatus splits into beams for each spin component S_z.

What, in this experiment, is essential to quantum measurement?

i) The measurement interaction lasts a limited time T. At all other times the atom and Stern-Gerlach apparatus are distinct, independent systems.

ii) It produces a change (the deflection of the atom) that corresponds to the value of the observable (S_z).

iii) It does not change the measured observable.

iv) In principle, the interaction time T can be very small (if $\partial B_z/\partial z$ is very large). Sometimes a fast measurement is preferable. For example, a measurement of a particle's position

[3] Since the field depends on z, it cannot be exactly parallel to the z-axis – it would not satisfy Maxwell's equation $\nabla \cdot \mathbf{B} = 0$. So the field must fringe away from the z-axis.

may change its momentum and thus, in time, its position; then, if the measurement is not fast, it measures a changed position.

v) The measurement is a quantum process. We treat the atom as a wave packet evolving along a trajectory through the magnetic field and beyond. We can do so only because we wrote down a Hamiltonian, H_{int}, for the measurement interaction.

If we accept i)–v) as criteria for quantum measurement, they lead us to a model (due to von Neumann [2]) for the measurement of an observable A_s. To satisfy v), we treat the measurement via an interaction Hamiltonian H_{int}. According to iii), H_{int} and A_s must commute. According to ii), H_{int} must couple A_s to something that yields an observable change, like the deflection of the atom in the measurement of S_z. The simplest coupling we can write down is $A_s P_d$, where P_d is an otherwise independent observable. Since i) requires H_{int} to be effective only during the measurement, we multiply $A_s P_d$ by a coupling $g(t)$ that is different from zero only in an interval $0 \leq t \leq T$, with

$$\int_{-\infty}^{\infty} g(t)dt = \int_{0}^{T} g(t)dt = g_0 \ . \tag{7.4}$$

(See Fig. 7.3.) Finally, iv) implies the limit $T \to 0$. In this limit the measurement is termed *impulsive*. Our interaction Hamiltonian is

$$H_{int}(t) = g(t)A_s P_d \ , \tag{7.5}$$

and the total Hamiltonian is

$$H = H_d + H_s + H_{int}(t) \ . \tag{7.6}$$

It includes the separate Hamiltonians of the measuring device (H_d) and of the measured system (H_s).

How does the model work? Because P_d is a quantum operator, some other quantum operator does not commute with it. Let the operator Q_d be conjugate to P_d:

$$[Q_d, P_d] = i\hbar \ .$$

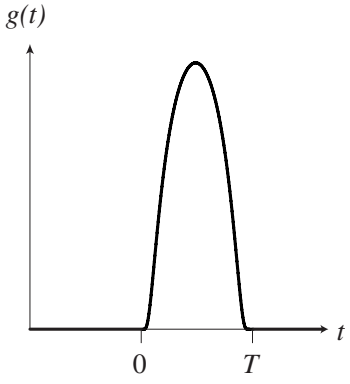

Figure 7.3: Graph of a possible coupling $g(t)$ in Eq. (7.4).

We prepare the measuring device in an initial state with well defined Q_d, say $Q_d(0) = 0$, and the measured system in a state with well defined A_s. According to the Heisenberg equation of motion, the change in Q_d during the measurement is

$$\begin{aligned} Q_d(T) - Q_d(0) &= \int_0^T dt \frac{dQ_d}{dt} \\ &= \int_0^T dt \frac{i}{\hbar}[H, Q_d] \\ &= \int_0^T dt \frac{i}{\hbar}[H_d, Q_d] + g_0 A_s \ . \end{aligned}$$

While $[Q_d, H_d]$ need not vanish, in the limit of small T the only term remaining in $Q_d(T)$ will be $g_0 A_s$. At the end of the measurement, we find the measuring device in a state with $Q_d = g_0 A_s$. (Note that Q_d, and not P_d, registers the result of the measurement, although it is P_d that appears with A_s in H_{int}.) We can choose g_0 large to get a macroscopic change in Q_d. It is convenient to identify Q_d with a pointer on a measuring device, indicating the measurement result on a dial.

So far, we have discussed quantum measurements in the Heisenberg formalism, in which an operator A_s evolves according to Heisenberg's equation of motion. An advantage of this formalism is that the classical and quantum equations of motion for A_s coincide, and we understand quantum evolution by analogy with classical evolution. We have so far assumed that the measured system is in an eigenstate of A_s, so we can replace the operator A_s by one of its eigenvalues. What if the measured system is not in an eigenstate of A_s? Chapter 9 addresses this question. (See also Prob. 7.2.)

As an application of the model, consider measuring the position of a particle via a local interaction with a probe particle. The interaction Hamiltonian is

$$H_{int}(t) = -g(t)\mathbf{x} \cdot \mathbf{X} \ ,$$

where $\mathbf{x}$ is the position of the particle we want to locate (corresponding to A_s in Eq. (7.5)) and $\mathbf{X}$ is the position of our probe (corresponding to P_d). Aside from the measurement interaction, the particles are free; their Hamiltonians are

$$H_s = \frac{p^2}{2m} \qquad H_d = \frac{P^2}{2M} \ ,$$

where $\mathbf{p}$ and $\mathbf{P}$ are the momenta conjugate to $\mathbf{x}$ and $\mathbf{X}$, respectively. Let $g(t)$ equal g_0/T for $0 \le t \le T$, and 0 otherwise, as in Fig. 7.4. Applying the Heisenberg equation of motion, we get

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{p}/m \\ \dot{\mathbf{p}} &= g(t)\mathbf{X} \\ \dot{\mathbf{X}} &= \mathbf{P}/M \\ \dot{\mathbf{P}} &= g(t)\mathbf{x} \ . \end{aligned} \tag{7.7}$$

All the dynamical quantities change during the measurement. (See Prob. 7.3.) However, the dynamics simplifies in the limit $T \to 0$. The change in the position coordinates becomes

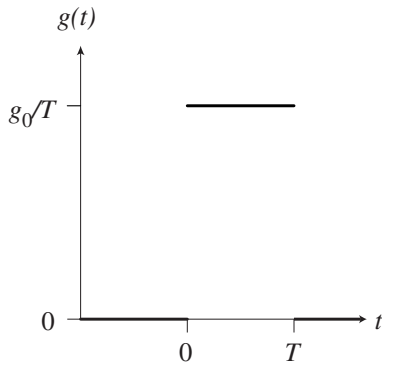

Figure 7.4: Graph of $g(t)$ in Eq. (7.7).

negligible, since

$$\mathbf{X}(T) - \mathbf{X}(0) = \frac{1}{M}\int_0^T \mathbf{P}dt \ ,$$
$$\mathbf{x}(T) - \mathbf{x}(0) = \frac{1}{m}\int_0^T \mathbf{p}dt \ ,$$

and if $\mathbf{P}$ and $\mathbf{p}$ remain finite for $T \to 0$, $\mathbf{X}$ and $\mathbf{x}$ remain constant in the same limit. We find then that $\mathbf{P}$ and $\mathbf{p}$ change by a finite amount,

$$\mathbf{P}(T) - \mathbf{P}(0) = g_0\mathbf{x} \ , \qquad \mathbf{p}(T) - \mathbf{p}(0) = g_0\mathbf{X} \ , \tag{7.8}$$

and the change in the momentum of the probe, $\mathbf{P}(T) - \mathbf{P}(0)$, yields the position $\mathbf{x}(0)$.

7.3 Quantum Measurements and Uncertainty Relations

Quantum measurements automatically obey the complementarity principle. Since the interaction Hamiltonian for measuring A_s contains A_s, measuring A_s disturbs any observable that does not commute with A_s. Quantum measurements also preserve the Heisenberg uncertainty relations. We can apply the von Neumann model to show that the uncertainty relations are consistent, i.e. no measurement can evade them.

Let us start with a special case: the uncertainty relation for position and momentum. Consider the position measurement of the last section. It is sufficient to consider one component of position, say x_1, and its conjugate momentum p_1. Since $\mathbf{p}(T) = \mathbf{p}(0) + g_0\mathbf{X}$, the position measurement increases the uncertainty in $p_1(T)$; even if $\Delta p_1(0)$ vanishes, we have

$$\Delta p_1(T) \geq g_0 \Delta X_1(0) \ . \tag{7.9}$$

We measure x_1 from the change in P_1/g_0, so the uncertainty in the measurement of x_1 cannot be less than the uncertainty in P_1/g_0; and $P_1(0)$ cannot be certain because the measuring device initially has a well defined pointer position, $X_1(0)$. Thus

$$\Delta x_1(T) \geq \Delta P_1(0)/g_0 \ . \tag{7.10}$$

By multiplying Eqs. (7.9–10) we find

$$\Delta x_1(T)\Delta p_1(T) \geq \Delta X_1(0)\Delta P_1(0) \ . \tag{7.11}$$

If the measuring device were classical, Eq. (7.11) would imply a violation of the uncertainty relation. But the measuring device is itself a quantum system and ΔX_1, ΔP_1 satisfy the uncertainty relation

$$\Delta X_1(0)\Delta P_1(0) \geq \hbar/2 \ . \tag{7.12}$$

It follows that

$$\Delta x_1(T)\Delta p_1(T) \geq \hbar/2 \ . \tag{7.13}$$

We assumed the uncertainty relation Eq. (7.12) for X_1 and P_1; however, we did not assume Eq. (7.13) for x_1 and p_1, we derived it from Eq. (7.12). Hence any measuring device satisfying Eq. (7.12) confirms Eq. (7.13).

We now generalize this proof to the case of two arbitrary observables A_s and B_s. We assume an interaction Hamiltonian for a measurement of A_s,

$$H_{int} = g(t)P_d A_s \ ,$$

where $g(t)$ satisfies Eq. (7.4). During the measurement, the conjugate variable Q_d evolves from $Q_d(0) = 0$ initially to $Q_d(T) = g_0 A_s$ at the end of the measurement. If we infer the value of A_s from the change in Q_d we have, for the uncertainty ΔA_s,

$$\Delta A_s = \Delta Q_d / g_0 \ .$$

We compute uncertainties with respect to the overall state of the measuring device and the measured system. Let us denote this state as $|\Psi_s, \Phi_d\rangle$,

$$|\Psi_s, \Phi_d\rangle = |\Psi_s\rangle \otimes |\Phi_d\rangle \ ,$$

where $|\Psi_s\rangle$ completely specifies the state of the measured system and $|\Phi_d\rangle$ completely specifies the state of the measuring device. It is a product state because the measuring device and measured system are uncorrelated initially. We define the expectation value of an operator O to be $\langle \Psi_s, \Phi_d | O | \Psi_s, \Phi_d \rangle$ and denote it $\langle O \rangle$.

During the measurement, any observable B_s of the measured system that does not commute with A_s also evolves. The Heisenberg equation of motion for B_s is

$$\frac{dB_s}{dt} = g(t)\frac{i}{\hbar}[A_s, B_s]P_d \ .$$

We take the expectation value of both sides of the equation. Since the expectation value of P_d vanishes (the pointer has no momentum initially, and P_d does not change during the measurement), there is no change in the expectation value of B_s or of $[A_s, B_s]$; however, the uncertainty in $\langle B_s \rangle$ is now at least

$$\Delta B_s = g_0 |\langle [A_s, B_s] \rangle| \Delta P_d / \hbar \ ,$$

and by multiplying ΔA_s and ΔB_s we obtain

$$\Delta A_s \Delta B_s \geq \Delta Q_d |\langle [A_s, B_s] \rangle| \Delta P_d/\hbar \geq \frac{1}{2} |\langle [A_s, B_s] | \rangle \ ,$$

since $\Delta Q_d \Delta P_d \geq \hbar/2$ by assumption. (Compare Prob. 3.10(b).) Quantum measurements preserve the uncertainty relations.

7.4 Paradox Lost

In Sects. 7.2–7.3 we have constructed quantum measuring devices to measure an observable (e.g. position or momentum) instantaneously, without changing it, and shown that they preserve the Heisenberg uncertainty relations. Let us now revisit the paradox of Sect. 7.1. What about a measurement of velocity?

A free particle moving in one dimension, with momentum p and mass m, has velocity $v = p/m$. We know how to measure p/m. If we apply a measurement interaction $H_{int} = g(t)P_d p/m$ to measure the velocity, however, the particle is no longer free, and the velocity is no longer p/m. The velocity is

$$v = \frac{dx}{dt} = \frac{i}{\hbar}[H, x] \ ;$$

if $H = p^2/2m + g(t)P_d p/m$, then $v = p/m + g(t)P_d/m$. Clearly, it does not help if we substitute $p/m + g(t)P_d/m$ for p/m in H_{int}; we merely find that velocity has again changed from what it was, and that our measuring device no longer measures velocity.

We arrived at a model of quantum measurement by assuming, with von Neumann, that coupling a measurement to an observable A_s does not change A_s. We now discover that we cannot couple a measurement to the velocity v without changing v. Must we discard the von Neumann model? No, the von Neumann model is correct as it applies to x and p and all other canonical quantities.[4] But, in general, it does not apply to *noncanonical* physical observables such as velocity and other time derivatives of canonical observables. A measurement of a noncanonical observable $\dot{A}_s$ in general changes $\dot{A}_s$. (See Probs. 7.8–7.9.)

The von Neumann model has no place for noncanonical observables because the Hamiltonian has no place for them. There is no direct way to write down a Hamiltonian with a coupling to a noncanonical observable such as v. However, the problem with measuring noncanonical observables extends beyond Hamiltonian mechanics. In Lagrangian mechanics it is straightforward to write down a coupling to v:

$$L = \frac{m}{2}\dot{x}^2 + \frac{M}{2}\dot{Q}_d^2 + g(t)\dot{x}Q_d \ . \tag{7.14}$$

The term $g(t)\dot{x}Q_d$ defines a measurement coupling directly to $v = \dot{x}$. If $g(t) = g_0/T$ for a time T, the change in $M\dot{Q}_d$ measures v. But again, v during the measurement is not what it would be if there were no measurement; the Lagrange equations of motion yield $v = v(0) - g(t)Q_d/m$,

[4]Canonical quantities are the generalized coordinates q_i and momenta p_i satisfying $\dot{q}_i = \partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial q_i$ for some function $H(q_i, p_i, t)$.

where $v(0)$ is the velocity at the beginning of the measurement. The Hamiltonian corresponding to L is

$$H = \frac{1}{2m}\left[p - g(t)Q_d\right]^2 + \frac{1}{2M}P_d^2$$

and the change in P_d indeed measures v, but v during the measurement is not p/m but $\left[p - g(t)Q_d\right]/m$.

The problem with the extra term $-g(t)Q_d/m$ in v is not that it prevents us from measuring velocity. The problem is that it contributes to the uncertainty in v. Let $\Delta Q_d(0)$ and $\Delta P_d(0)$ denote the uncertainties in Q_d and P_d, respectively, at the beginning of the measurement. If $\Delta Q_d(0)$ is small, then $\Delta P_d(0)$ is large. But the measured value of v is $[P_d(T) - P_d(0)]/g_0$; the uncertainty in the velocity measurement depends both on $P_d(0)$ and, through $P_d(T)$, on $Q_d(0)$. Minimizing $\Delta Q_d(0)$ conflicts with minimizing $\Delta P_d(0)$, and so there is a minimum uncertainty in the measured value of v:

$$\Delta v \geq \sqrt{\hbar/mT}\ . \tag{7.15}$$

(See Prob. 7.10.) Equation (7.15) resolves the velocity paradox of Sect. 7.1. Now the uncertainty in x at time T is at least $T\Delta v$ while the uncertainty in p at time T is at least $m\Delta v$, i.e.

$$\Delta x(T) \cdot \Delta p(T) \geq (T\Delta v)(m\Delta v) \geq \hbar\ ,$$

consistent with Heisenberg uncertainty relation. However, Eq. (7.15) can be misleading. Beware! Our discussion of quantum measurements is not complete.[5] The next chapter completes the treatment of noncanonical observables.

Problems

7.1 Show that if there exists a Hermitian operator T such that $[T, H] = i\hbar$, then the expectation value of $H = p^2/2m$ can be negative.

7.2 For an impulsive measurement with H_{int} of Eq. (7.5) as its Hamiltonian, show that $\langle Q_d(T)\rangle - \langle Q_d(0)\rangle = g_0\langle A_s\rangle$, where the $\langle A_s\rangle$ represents the expectation value of A_s in the initial state.

7.3 Compute $P(T) - P(0)$ exactly from Eq. (7.7) (for T nonzero) and verify Eq. (7.8) in the limit $T \to 0$.

7.4 Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{P_1^2}{2M_1} + \frac{P_2^2}{2M_2} + g(t)[xX_1 + \alpha pX_2]\ ,$$

which represents two measuring devices (subscripts 1 and 2) simultaneously measuring the position x and momentum p of a third system. (The constant α has appropriate units.) Show that the measurements preserve the uncertainty relation $\Delta x \Delta p \geq \hbar/2$.

[5] If we measure p/m via the Hamiltonian $H = p^2/2m + g(t)P_d p/m$, for how long does v not equal p/m?

7.5 Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{P_1^2}{2M_1} + \frac{P_2^2}{2M_2} + g_1(t)xX_1 + \alpha g_2(t)pX_2 \ ,$$

which, as in the previous problem, represents two measuring devices (subscripts 1 and 2), but here $g_1(t)g_2(t) = 0$ (with α constant). Show that the measurements preserve the uncertainty relation $\Delta x \Delta p \geq \hbar/2$.

*7.6 Define

$$H = B_z L_z + b_z l_z + g(t) L_x l_x \ ,$$

where $\mathbf{L}$ and $\mathbf{l}$ are the angular momenta of the measuring device and measured system, respectively, and B_z, b_z are magnetic fields. Show that this Hamiltonian generates a measurement of l_x, and that if $l_x = l_y = l_z = 0$ initially, the measurement of l_x leaves $l_x = l_y = l_z = 0$ in the final state.

7.7 Consider the Hamiltonian of Prob. 7.5, taking $g_1(t) = g_2(t) = 1/T$ for $0 \leq t \leq T \ll 1$ (the measurement is impulsive) and $g_1(t) = g_2(t) = 0$ otherwise. What do the changes in P_1 and P_2 measure?

*7.8 Consider an interaction Hamiltonian

$$H_{int} = g(t)\frac{i}{2\hbar}\left([H,x]P_d + P_d[H,x]\right) \ , \tag{7.16}$$

where H is the *total* Hamiltonian $p^2/2m + H_d + H_{int}$ so that $i[H,x]/\hbar = \dot{x}$. (Thus we define H and H_{int} in Eq. (7.6) implicitly.) We have written the product of $[H,x]$ and P_d symmetrically because $[H,x]$ could in principle contain terms not commuting with P_d. The coupling $g(t)$ obeys Eq. (7.4) for arbitrarily short T. Compute the change in Q_d and show that $Q_d(T) - Q_d(0) \neq ig_0 \int_0^T [H,x]dt/\hbar$, so the interaction Hamiltonian of Eq. (7.16) does not measure $\dot{x}$.

7.9 Show that if q and p are canonically conjugate observables, a quantum measurement of $\dot{q}$ (according to Eq. (7.5) with $i[H_s,q]/\hbar$ in the place of A_s) must change $\dot{q}$ unless $\partial^2 H_s/\partial p^2 = 0$.

7.10 Prove Eq. (7.15), taking $g(t)$ constant over the interval $0 \leq t \leq T$.

References

[1] See, for example, J. D. Jackson, *Classical Electrodynamics*, Second Edition (New York: Wiley), 1975, p. 186.

[2] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, trans. R. T. Beyer (Princeton: Princeton U. Press), 1955; Chaps. V-VI reprinted in WZ pp. 549–647.

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8 Measurement and Compensation

The last chapter presents von Neumann's model for measurements of canonical quantities. The model has many applications, and Sect. 7.4 applies it to resolve the velocity paradox of Sect. 7.1. However, we may not be comfortable with the resolution of the paradox. Velocity is not a canonical quantity, but it is gauge invariant: $m\mathbf{v} = \mathbf{p} - e\mathbf{A}/c$. (See Prob. 4.3.) The model implies that a velocity measurement lasting a time T yields v with uncertainty $\Delta v \geq \sqrt{\hbar/mT}$. (See Prob. 7.10.) Yet the same model implies that we can measure momentum instantaneously, with no minimum uncertainty, and without changing it. We can also measure $\mathbf{p} - e\mathbf{A}/c$ instantaneously, with no minimum certainty, and without changing it; $\mathbf{A}$ is a canonical variable. But $\mathbf{p} - e\mathbf{A}/c$ equals $m\mathbf{v}$ (except during the measurement) so why can't we measure velocity instantaneously? Likewise, in the laboratory we measure electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, not the potentials $\mathbf{A}$ and V that appear in the Hamiltonian, Eq. (4.7). The magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is a canonical quantity, but the electric field is not (although it is gauge invariant), because it depends on the derivative of $\mathbf{A}$ with respect to time: $\mathbf{E} = -(1/c)(\partial\mathbf{A}/\partial t) - \nabla V$. Why can't we measure $\mathbf{E}$ instantaneously?

The question of measuring $\mathbf{E}$ troubled such physicists as Landau, Peierls, Bohr and Rosenfeld. Rosenfeld arrived at Bohr's institute in early 1931 just as the question had come to a head. He ran into Gamow and asked what was new. Gamow answered with a drawing he had just made. It showed Landau, tightly bound to a chair and gagged, while Bohr stood before him with upraised finger, saying "Please, please, Landau, can I just get a word in!" Landau and Peierls had come a few days before with a new paper to show Bohr, "but he does not seem to agree," said Gamow airily, "and this is the kind of discussion which has been going on all the time." Peierls had left the previous day, "in a state of complete exhaustion," added Gamow. Landau stayed on for a few weeks, and Rosenfeld discovered that "Gamow's representation of the situation was only exaggerated to the extent usually conceded to artistic fantasy" [1]. Landau and Peierls had considered measuring the electric field by sending a charged test particle through it [2]. The electric field deflects the test charge; the change in the momentum of the charge indicates the field strength. But an accelerated charge radiates and loses an unknown part of its momentum to the electromagnetic field. (See Prob. 8.1.) We can suppress the radiation by reducing the charge on the test particle; then the momentum of the particle changes more slowly, but the measurement lasts longer. An impulsive, accurate measurement of the electric field is impossible. Bohr felt uneasy, but could not refute the claim of Landau and Peierls.

Ultimately, Bohr and Rosenfeld did refute the claim; it took them nearly three years [3], but they showed how to measure electric and magnetic fields instantaneously [4]. Bohr and Rosenfeld did not treat noncanonical observables in general. We will see that in the von

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Neumann model it is straightforward to measure any noncanonical observable instantaneously; we only have to *apply* the model differently to noncanonical observables. In the next section we consider measuring energy, including kinetic energy, in von Neumann's model. Kinetic energy is a noncanonical observable, since it depends on the velocity $\mathbf{v}$. An application of the uncertainty relation for energy and time indicates that we cannot measure E instantaneously. The paradox is that we can!

8.1 Paradox Regained

Numerous papers and books claim that a measurement of energy cannot take an arbitrarily short time. They interpret the energy-time uncertainty relation, Eq. (2.10), as follows: the faster the energy measurement, the more uncertain the result. Let us examine some arguments for this interpretation.

i) A simple argument starts with Einstein's equation relating energy and frequency, Eq. (2.3): $E = h\nu$. Suppose a quantum wave takes a time T to pass through a measuring device. Since the wave lasts a time T, its Fourier transform is large for frequencies in a range that includes $0 \le \nu \le 1/T$. Then $\Delta E = h\Delta\nu \ge h/T$.

ii) Suppose we have a system in its ground state of energy E_0. We want to prepare it in a final state of energy E, with E in its continuous energy spectrum. We apply a perturbation that oscillates in time with angular frequency $\omega = (E - E_0)/\hbar$, lasting a time T. That is, it takes us time T to measure (prepare) the final state of energy E. But the perturbation might excite the system to a state of a different energy E' instead. The relative probability of a transition to a final state of energy E' is approximately

$$\frac{\sin^2\left[(E' - E_0)T/2\hbar - \omega T/2\right]}{\left[(E' - E_0) - \hbar\omega\right]^2} = \frac{\sin^2\left[(E' - E)T/2\hbar\right]}{(E' - E)^2} \,. \tag{8.1}$$

(See Prob. 6.11.) Figure 8.1 shows a graph of Eq. (8.1) as a function of $E' - E$. The dominant peak, lying between $-h/T$ and h/T, contains over 90% of the area, so with probability 90% the final energy of the system will lie between $E - h/T$ and $E + h/T$, i.e. $\Delta E \ge h/T$ again.

iii) We may measure the kinetic energy of a particle by hitting it with a probe particle. Consider a particle of initial energy e_i and momentum p_i in an elastic collision with a probe particle of initial energy E_i and momentum P_i. For simplicity, let the particles have equal masses m and move on a line. Since the collision conserves total energy and momentum, the final kinetic energies of the probe particle and measured particle are $E_f = e_i$ and $e_f = E_i$, respectively. And since $E_f = P_f^2/2m$ and $E_i = P_i^2/2m$, we measure e_i or e_f by measuring P_f or P_i, respectively. Hence a measurement of either e_i or e_f can be impulsive and accurate. But the more accurately we measure P_f or P_i, the greater the uncertainty in X_f or X_i, the respective position of the probe. Without knowing where the probe particle is, we cannot know the time of the collision. Then how long does the measurement take? If we measure P_i, we prepare a state of energy $e_f = P_i^2/2m$ after the collision; but we must await the collision a time T not less than ΔX_i divided by the initial speed of the probe:

$$T \ge \frac{\Delta X_i}{P_i/m} \ge \frac{\hbar}{2P_i\Delta P_i/m} = \frac{\hbar}{2\Delta E_i} = \frac{\hbar}{2\Delta e_f} \,. \tag{8.2}$$

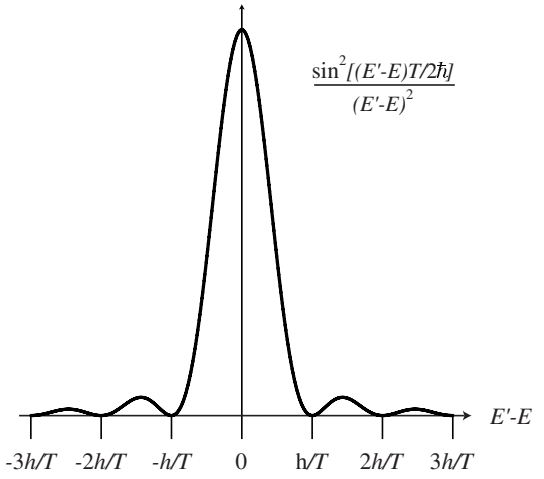

Figure 8.1: Graph of Eq. (8.1).

(See Prob. 3.10.) Thus, to measure (prepare) the state of energy e_f takes a time $T \geq \hbar/2\Delta e_f$. What if we measure P_f? We must measure P_f *after* the collision to obtain e_i *before* the collision. Even if ΔX_i vanishes, Δx_i does not, and we must wait a time

$$T \geq \frac{\Delta x_i}{p_i/m} \geq \frac{\hbar}{2p_i\Delta p_i/m} = \frac{\hbar}{2\Delta e_i} \tag{8.3}$$

for the collision. Thus a measurement of e_i lasts a time $T \geq \hbar/2\Delta e_i$.

iv) We can apply Eq. (7.15), which states that if a measurement of velocity v takes a time T, the uncertainty in v during the measurement is $\Delta v \geq \sqrt{\hbar/mT}$. The uncertainty in the kinetic energy, $E = mv^2/2$, during the measurement is $\Delta E = mv\Delta v$. Thus

$$(\Delta E)^2 = m^2v^2(\Delta v)^2 \geq m^2v^2(\hbar/mT) = 2E\hbar/T \; . \tag{8.4}$$

To measure E (and not just bound E from above) we need $\Delta E \leq E$. Thus Eq. (8.4) implies

$$\Delta E \geq 2\hbar/T \; . \tag{8.5}$$

Arguments i) – iv) suggest that there is no way to measure energy impulsively. From Eqs. (8.2–3) and (8.5) we conclude

$$T\Delta E \geq \hbar/2 \; , \tag{8.6}$$

with the interpretation that T is the time it takes to measure E (whether E is the energy before, during, or after the measurement).

Nevertheless, there are arguments against this interpretation. The Heisenberg uncertainty relations for two operators A, B follow directly from Eq. (4.15), $\Delta A\Delta B \geq \frac{1}{2}|\langle[A, B]\rangle|$. But time is not an operator in nonrelativistic quantum mechanics; there is no operator T such that $[T, E] = i\hbar$. (See Sect. 7.1 and Prob. 7.1.) Hence Eq. (8.6) remains unproved. The measurements in i) – iv) obey Eq. (8.6), but other measurements may not. Also, Eq. (8.6) is an odd exception to the Heisenberg uncertainty relations. The other uncertainty relations refer to

two properties of a measured system – position and momentum, etc. Eq. (8.6) refers, instead, to a property of the measured system (energy) and a property of the measuring device (how long the measurement lasts). An analogous interpretation of $\Delta p \Delta x \geq \hbar/2$ would be the following: a device to measure momentum with accuracy Δp can be no smaller than $\Delta x \geq \hbar/2\Delta p$. This interpretation is false! For example, a very heavy atom in its ground state is a device to measure photon momentum. If a photon excites the atom to a state with lifetime T, the uncertainty in the photon energy is $\Delta E \geq \hbar/T$; hence the uncertainty in its momentum is $\Delta p = \Delta E/c \geq \hbar/cT$. The time the photon leaves the atom is uncertain, with uncertainty T, so $\Delta x \approx cT$ and $\Delta x \Delta p \geq \hbar$. But the size of the atom may be much less than cT.

Now consider the following interaction Hamiltonian for measuring energy:

$$H_{int}(t) = g(t) H_s P_d \ ,$$

where the coupling $g(t)$ obeys $\int_0^T g(t)dt = g_0$. As in Eq. (7.6), the total Hamiltonian for the measuring device and the measured system is

$$H = H_d + H_s + H_{int}(t) \ .$$

As long as H_{int} is nonzero, we cannot identify H_s with the energy of the measured system. But the equation of motion for Q_d,

$$\frac{dQ_d}{dt} = \frac{i}{\hbar}[H, Q_d] = \frac{i}{\hbar}[H_d, Q_d] + g(t) H_s \ ,$$

implies that the total change in Q_d during the measurement is

$$Q_d(T) - Q_d(0) = \frac{i}{\hbar} \int_0^T dt [H_d, Q_d] + g_0 H_s \ ; \tag{8.7}$$

the second term on the right side of Eq. (8.7), divided by g_0, is the energy of the system before and after the measurement. Moreover, the time T is arbitrary; even if $[Q_d, H_d]$ is nonzero, for T small we obtain $g_0 H_s$ as the change in Q_d during the measurement. We measure the energy of the system in an arbitrarily short time, and the final energy is unchanged from the initial energy!

This conclusion [5] contradicts Eq. (8.6) and its interpretation. If it is correct, then H_{int} must represent a measurement we have so far not considered. What is this measurement?

8.2 Compensating Forces

Section 7.4 describes a measurement of velocity v according to the interaction Hamiltonian

$$H_{int} = g(t) \frac{p}{m} P_d \ , \tag{8.8}$$

and shows that measuring v changes v. Yet if we calculate $Q_d(T) - Q_d(0)$ with this interaction Hamiltonian, we find

$$Q_d(T) - Q_d(0) = \frac{i}{\hbar} \int_0^T dt [H_d, Q_d] + g_0 \frac{p}{m} \ .$$

Even if $[H_d, Q_d] \neq 0$, for T small we can neglect the integral; then Q_d measures p/m, a constant of the motion equal to v before and after the measurement. This velocity measurement is analogous to the energy measurement at the end of the last section. What measurement does it represent?

We sometimes gain insight into a quantum problem by considering – with great care – the corresponding classical problem. The classical problem of measuring velocity (unlike the quantum problem) allows us to start with a Lagrangian, in which velocities appear explicitly. Suppose we try to measure the velocity $\mathbf{v}$ of a particle of mass m via a probe particle of mass M. A possible Lagrangian is

$$L = \frac{1}{2}MV^2 + \frac{1}{2}mv^2 + g(t)\mathbf{v}\cdot\mathbf{X} \; .$$

(Compare Eq. (7.14).) L contains, besides the kinetic energies of the two particles, an interaction term that couples $\mathbf{v}$ to the position $\mathbf{X}$ of the probe. This term resembles the interaction Hamiltonian of Eq. (8.8). L leads to the following equations of motion:

$$\frac{d}{dt}(M\mathbf{V}) = g(t)\mathbf{v} \tag{8.9a}$$

$$\frac{d}{dt}\left[m\mathbf{v} + g(t)\mathbf{X}\right] = 0 \; . \tag{8.9b}$$

According to Eq. (8.9a), the change in $M\mathbf{V}$ during the measurement is exactly $\mathbf{v}$. If $\mathbf{v}$ were a constant of the motion, Eq. (8.9a) would represent a measurement of $\mathbf{v}$. However, Eq. (8.9b) tells us that $\mathbf{v}$ is not a constant of the motion – it changes drastically, although it returns to its original value at the end of the measurement. Even so, we have solved the classical problem, because classical physics allows us to measure $g(t)\mathbf{X}/m$, the change in $\mathbf{v}$ during the measurement, as well as $M\mathbf{V}$. But we have not solved the quantum problem, because quantum physics does not allow us to measure both $\mathbf{X}$ and $M\mathbf{V}$, and we need both to obtain $\mathbf{v}$.

We might try to change Eq. (8.9b) to make $\mathbf{v}$ a constant of the motion. But $\mathbf{v}$ cannot be a constant of the motion if it couples to anything. On the other hand, we can try changing Eq. (8.9a) so that $dM\mathbf{V}/dt$ equals $\mathbf{v} + g(t)\mathbf{X}/m$, which *is* a constant of the motion. By adding a term to L we get a Lagrangian L'

$$\begin{aligned} L' &= L + \frac{1}{2m}\left[\mathbf{X}g(t)\right]^2 \\ &= \frac{1}{2}MV^2 + \frac{1}{2}mv^2 + g(t)\mathbf{v}\cdot\mathbf{X} + \frac{1}{2m}\left[\mathbf{X}g(t)\right]^2 \; , \end{aligned}$$

leading to the equations of motion

$$\frac{d}{dt}(M\mathbf{V}) = g(t)\left[\mathbf{v} + g(t)\mathbf{X}/m\right] \tag{8.10a}$$

$$\frac{d}{dt}\left[m\mathbf{v} + g(t)\mathbf{X}\right] = 0 \; . \tag{8.10b}$$

Now $dM\mathbf{V}/dt$ equals $\mathbf{v} + g(t)\mathbf{X}/m$, still a constant of the motion since Eqs. (8.9b) and (8.10b) are identical. The total change in $M\mathbf{V}$ during the measurement equals the velocity $\mathbf{v}$ before and after the measurement. Thus, we can measure $\mathbf{v}$ by arranging for an additional force $[g(t)]^2\mathbf{X}/m$ to act on the probe; this force compensates for the uncontrolled change in $\mathbf{v}$ during the measurement by inducing the same change in $M\mathbf{V}$, so the total change in $M\mathbf{V}$ equals $\mathbf{v}$.

The force is like a time-dependent spring acting on the probe particle. Since it depends on $\mathbf{X}$ it is totally uncertain; however, we don't need to know what it is in order to measure $\mathbf{v}$.

The Hamiltonian corresponding to L' is

$$\begin{aligned} H &= \frac{1}{2M}P^2 + \frac{1}{2m}\left[\mathbf{p} - g(t)\mathbf{X}\right]^2 - \frac{1}{2m}\left[\mathbf{X}g(t)\right]^2 \\ &= \frac{P^2}{2M} + \frac{p^2}{2m} - g(t)\frac{\mathbf{p}}{m}\cdot\mathbf{X}\ ; \end{aligned} \tag{8.11}$$

it is identical to the Hamiltonian for measuring position in Sect. 7.2 except that $\mathbf{p}/m$ has taken the place of $\mathbf{x}$ in the interaction term. The two equivalent expressions for H in Eq. (8.11) show that a measurement of $\mathbf{v}$, including the compensating force on the probe, is identical to a measurement of $\mathbf{p}/m$. The change in $\mathbf{P}$ during the measurement yields $\mathbf{p}/m$, which is the velocity before and after the measurement. (See also Prob. 8.4.) So measuring velocity is indeed as simple as measuring momentum.

8.3 Quantum Measurements of Noncanonical Observables

It is straightforward to generalize this velocity measurement; once we know how to measure velocity, we know how to measure any noncanonical variable. Let A denote a noncanonical observable of a system before, during and after a measurement. In general, the definition of A involves H, the total Hamiltonian of the system and measuring device, including an interaction Hamiltonian, as well as canonical observables of the system and their commutators with H. Because A is a noncanonical observable, it changes during the measurement. Let A_s denote the same observable *before* and *after* a measurement. The difference in the definition of A_s and A is the following: everywhere that H appears in the definition of A, H_s takes its place in the definition of A_s. For example, if $A = [H, O]$ for some operator O, then $A_s = [H_s, O]$. Yet the prescription for measuring a noncanonical variable is to put A_s, and not A, in the interaction Hamiltonian:

$$H_{int}(t) = g(t)A_sP_d\ .$$

Although A_s differs from A during the measurement, at the beginning and end of the measurement $A = A_s$. We now summarize the general treatment of noncanonical observables as follows:

i) The total Hamiltonian for the measurement is $H = H_s + H_d + g(t)A_sP_d$. In general, H entails compensating forces on the measuring device.

ii) The value of the noncanonical observable is the same before and after the measurement, although its definition changes during the measurement.

iii) The measurement yields the value of the noncanonical observable before and after, not during, the measurement.

So, if we only interpret it properly, von Neumann's model extends to measurements of noncanonical variables. In the next section we apply the model to the problem of measuring the electric field impulsively (one of the problems that Bohr and Rosenfeld solved).

For any noncanonical observable, H_{int} entails compensating forces on the measuring device. The Hamiltonian often hides these compensating forces, but they appear in the equations of motion for the noncanonical observables. There may also be compensating forces in measurements of *canonical* observables. For example, Sect. 7.2 discusses a measurement of the position of a particle using a probe particle and the interaction Hamiltonian $H_{int} = -g(t)\mathbf{x}\cdot\mathbf{X}$. This measurement yields $\mathbf{x}$ in the limit of an impulsive measurement. However, we can obtain the average value of $\mathbf{x}$ during a time T from a measurement that is not impulsive. The Hamiltonian for such a measurement contains, in addition to $H_{int}(t)$, a term representing compensating forces on the probe particle. The compensating forces depend on H_s. (See Probs. 8.6 and 8.7.)

How do we know when a nonimpulsive measurement of A_s, a canonical variable, requires compensating forces? Consider the commutator $[[A_s, H], A_s]$. It contains the commutator $[A_s, H]$, which is proportional to $\dot{A}_s$. If $\dot{A}_s$ does not commute with A_s, i.e. if $[[A_s, H], A_s] \neq 0$, then $\dot{A}_s$ does not commute with the measurement interaction $H_{int}(t) = g(t)A_sP_d$; thus the measurement changes the evolution of $A_s(t)$ from what it would have been in the absence of the measurement, and the measurement requires compensating forces.

8.4 Measuring the Electric Field

In this treatment of noncanonical observables, we do not have to invent compensating forces for each new measurement; the compensating forces come out automatically. Thus the derivation of measurement interactions is simpler. To illustrate the treatment, we can show how to measure the electric field impulsively [6], as did Bohr and Rosenfeld.

Suppose we want to measure E_x. Following Bohr and Rosenfeld, we measure $E_x(\mathbf{x})$ folded with a rigid charge distribution: our measuring device is a heavy, charged sphere, of diameter D, in the electric field. The sphere is free to move only in the x-direction; let X denote the coordinate of its center of mass. The Hamiltonian for the measuring device is

$$H_d = \frac{P^2}{2M} \ ,$$

with $[X, P] = i\hbar$. We treat the electromagnetic field in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. In this gauge, the scalar potential $A_0 = V$ is the instantaneous Coulomb potential arising from any charges; it decouples from the radiation field [7]. (See Probs. 8.8–9.) The Hamiltonian for the radiation field is a sum over modes with propagation vectors $\mathbf{k}$ and angular frequencies $\omega = ck = c|\mathbf{k}|$:

$$H_{rad} = \frac{1}{2}\sum_{\mathbf{k}} \left[\mathbf{\Pi}_{\mathbf{k}} \cdot \mathbf{\Pi}_{-\mathbf{k}} + \omega^2 \mathbf{A}_{\mathbf{k}} \cdot \mathbf{A}_{-\mathbf{k}}\right] \ ; \tag{8.12}$$

here $\mathbf{\Pi}_{\mathbf{k}} = d\mathbf{A}_{-\mathbf{k}}/dt$ is the momentum conjugate to the field coordinate $\mathbf{A}_{\mathbf{k}}$ and $[A_{\mathbf{k},i}, \Pi_{\mathbf{k}',j}] = i\hbar(\delta_{ij} - k_ik_j/k^2)\delta_{\mathbf{k},\mathbf{k}'}$. (See Prob. 8.11.) The commutation relations are unusual because $A_{\mathbf{k},1}$, $A_{\mathbf{k},2}$ and $A_{\mathbf{k},3}$ are not all independent degrees of freedom: the gauge condition implies $\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}} = 0$, so for each wave vector $\mathbf{k}$ there are only two degrees of freedom, $\epsilon_1 \cdot \mathbf{A}_{\mathbf{k}}$ and $\epsilon_2 \cdot \mathbf{A}_{\mathbf{k}}$, where ϵ_1 and ϵ_2 are unit vectors orthogonal to $\mathbf{k}$. Equivalently, we can write

$$[\epsilon_i \cdot \mathbf{A}_{\mathbf{k}}, \epsilon_j \cdot \mathbf{\Pi}_{\mathbf{k}'}] = i\hbar\, \delta_{ij}\delta_{\mathbf{k},\mathbf{k}'} \ .$$

For simplicity, we assume that there are no charges aside from the test charge. Then $V = 0$ and the electric field is proportional to the canonical momentum; we have[1]

$$\mathbf{\Pi}(\mathbf{x}) = \frac{1}{4\pi c^2}\frac{d}{dt}\mathbf{A}(\mathbf{x}) = -\frac{1}{4\pi c}\mathbf{E}(\mathbf{x}) \ .$$

Then for measuring $E_x(\mathbf{x})$ our prescription tells us to couple the measuring device to the momentum $\Pi_x(\mathbf{x})$. The coupling includes the charge distribution $\rho(\mathbf{x})$ of the sphere, since we measure E_x folded with this rigid charge distribution. So the interaction Hamiltonian is

$$H_{int} = -g(t)X\int d^3x \Pi_x(\mathbf{x})\rho(\mathbf{x}) = -g(t)X\sum_{\mathbf{k}}\rho_{\mathbf{k}}\Pi_{\mathbf{k},x} \ , \tag{8.13}$$

where the $\rho_{\mathbf{k}}$ are the Fourier components of the charge distribution, and $g(t)$ is a constant g_0/T during an interval $0 \le t \le T$. The equations of motion for $H = H_d + H_{rad} + H_{int}$ are

$$\begin{aligned}
\frac{dX}{dt} &= \frac{P}{M} \\
\frac{dP}{dt} &= g(t)\sum_{\mathbf{k}}\rho_{\mathbf{k}}\Pi_{\mathbf{k},x} \\
\frac{d\mathbf{A}_{\mathbf{k}}}{dt} &= \mathbf{\Pi}_{-\mathbf{k}} - g(t)X\rho_{\mathbf{k}}\left[\epsilon_{1,x}\epsilon_1 + \epsilon_{2,x}\epsilon_2\right] \\
\frac{d\mathbf{\Pi}_{\mathbf{k}}}{dt} &= -\omega^2\mathbf{A}_{-\mathbf{k}} \ ,
\end{aligned} \tag{8.14}$$

where $\epsilon_{1,x}$ and $\epsilon_{2,x}$ are the x-components of ϵ_1 and ϵ_2, respectively. (See also Prob. 8.12.) In the impulsive limit, X and $\Pi_{\mathbf{k},x}$ do not change (since P and $A_{-\mathbf{k},x}$ remain finite). Thus the change in P over the course of the measurement is $g_0\sum_{\mathbf{k}}\rho_{\mathbf{k}}\Pi_{\mathbf{k},x}$. Now E_x changes during the measurement (although Π_x does not) but before and afterwards it is proportional to Π_x, so the change in P measures $\int d^3x E_x(\mathbf{x})\rho(\mathbf{x})$ before and after the measurement.

To find out what compensating forces act on the charge, we can compute the Lagrangian corresponding to H:

$$\begin{aligned}
L = \frac{M}{2}\left(\frac{dX}{dt}\right)^2 &+ \frac{1}{2}\sum_{\mathbf{k}}\left[\frac{d}{dt}\mathbf{A}_{\mathbf{k}}\cdot\frac{d}{dt}\mathbf{A}_{-\mathbf{k}} - \omega^2\mathbf{A}_{\mathbf{k}}\cdot\mathbf{A}_{-\mathbf{k}}\right] \\
&+ g(t)X\sum_{\mathbf{k}}\rho_{-\mathbf{k}}\frac{dA_{\mathbf{k},x}}{dt} + \frac{1}{2}\left[g(t)X\right]^2\sum_{\mathbf{k}}\rho_{\mathbf{k}}\rho_{-\mathbf{k}}[1 - k_x^2/k^2] \ .
\end{aligned}$$

From L we can see that the compensating force on the charge must be harmonic with a negative spring constant κ:

$$\kappa = -\left[g(t)\right]^2\sum_{\mathbf{k}}\rho_{\mathbf{k}}\rho_{-\mathbf{k}}[1 - k_x^2/k^2] \ . \tag{8.15}$$

If the measurement is not impulsive but lasts a time T, we require an additional compensating force, and κ changes to

$$\kappa = -\left[g(t)\right]^2\sum_{\mathbf{k}}\rho_{\mathbf{k}}\rho_{-\mathbf{k}}[1 - k_x^2/k^2]\frac{\sin\omega T}{\omega T} \ . \tag{8.16}$$

[1] In this book (as in Jackson, *op. cit.*) we use Gaussian units.

Then the change in P measures what the average value of $\int d^3x E_x(\mathbf{x})\rho(\mathbf{x})$ would have been in the absence of a measurement. (See Prob. 8.13.) From Eq. (8.16) we learn what *impulsive* means for this measurement. We assume that $\mathbf{\Pi_k}$ does not change during the measurement (since T is short and $\mathbf{A_{-k}}$ is finite). But this assumption is not consistent with Eq. (8.14) if ω is arbitrarily large. It is consistent with Eq. (8.14) for $\omega T \ll 1$ (as we see by combining the equations of motion for $\mathbf{\Pi_k}$ and $\mathbf{A_{-k}}$) and, indeed, the condition $\omega T \ll 1$ reduces Eq. (8.16) to Eq. (8.15). When does this condition hold? The measuring device cannot couple to modes with wavelengths that are much smaller than the size D of the rigid charge distribution, i.e. to frequencies that are much larger than c/D. The decoupling of frequencies greater than about c/D implies the condition $\omega T \ll 1$ if $D \gg cT$. Thus if $D \gg cT$, the measurement is impulsive.

As noted in the last section, measurements of canonical quantities, too, may require compensating forces. In particular, measurements of canonical fields – including the magnetic field – may require compensating forces. Every measurement, even an impulsive measurement, lasts a time $T > 0$. But for any T, there are modes with periods much shorter than T. If the measurement couples to these modes, it will yield time-averaged values and require compensating forces.

8.5 Energy and Time

Section 8.2 indicates that it is an error to interpret ΔT in

$$\Delta E \Delta T \geq \hbar/2 \tag{8.17}$$

as the time it takes to measure energy with uncertainty ΔE. Sect. 8.2 also suggests the source of the error: If a measurement lasts a time ΔT, then the uncertainty in the energy *during the measurement* satisfies Eq. (8.17). However, the uncertainty in the energy before and after the measurement need not satisfy Eq. (8.17). On the other hand, an uncertainty relation for energy and time is at the heart of the clock-in-the-box paradox of Sect. 2.4. How do we interpret it?

As noted in Sect. 8.1, the other Heisenberg uncertainty relations refer to two properties of a measured system, such as position and momentum. To conform to them, Eq. (8.17) must refer to the time of the measured system, not of the measuring device. What defines the time of the measured system? An event – for example, a quantum jump – defines a time of the system. So does an observable that changes smoothly in time – its evolution measures time. We call this time the *internal* time of the system to distinguish it from the *external* time of the measuring device. If we review the clock-in-the-box paradox as well as arguments i)–iv) for Eq. (8.6) and its interpretation, we find they are compatible with Eq. (8.17) if T refers to internal time.

It is not hard to show rigorously that this interpretation is correct. Let A_s be some observable of a system that changes in time. For example, A_s could be the position of a particle. We define the internal time of the system to be the operator

$$T = \frac{A_s}{\langle dA_s/dt \rangle} = \hbar \frac{A_s}{|\langle [A_s, H_s] \rangle|} \, ,$$

where H_s is the Hamiltonian of the system. The uncertainty in this time is

$$\Delta T = \hbar \frac{\Delta A_s}{|\langle [A_s, H_s] \rangle|} \ . \tag{8.18}$$

For example, if a particle moves at constant speed v, the uncertainty in the time it defines by its motion is $\Delta x/v$. If we multiply Eq. (8.18) on both sides by the uncertainty in the energy of the system, $\Delta E = \Delta H_s$, we have

$$\Delta T \Delta E = \hbar \frac{\Delta A_s \Delta H_s}{|\langle [A_s, H_s] \rangle|} \ .$$

Equation (4.15) implies that $\Delta A_s \Delta H_s \geq |\langle [A_s, H_s] \rangle|/2$, so Eq. (8.17) follows.

Sections 8.1–2 show how to measure the energy of a system accurately in arbitrarily short external time. These sections tacitly assume that the Hamiltonian H_s of the system *is already known*. If the Hamiltonian is not known, neither is the interaction Hamiltonian $H_{int} = g(t) H_s P_d$ assumed for the energy measurement; then the conclusions of Sects. 8.1–2 do not apply, and a measurement of energy to accuracy ΔE does require [8] a minimum time ΔT, where $\Delta E \Delta T \approx \hbar$. Indeed, even if we think we know the Hamiltonian H_s, we may be mistaken; then an arbitrarily short measurement would not yield the correct energy and we would also not discover our mistake.

This minimum time ΔT is internal time. The internal time T, unlike the external time, depends on H_s. A measurement of energy requires a minimum time ΔT because a system with Hamiltonian H_s evolves in time ΔT according to the unitary transformation $e^{-iH_s \Delta T/\hbar} \approx 1 - iH_s \Delta T/\hbar$. If H_s is unknown, we estimate it from the second term in this expansion, which is proportional to ΔT. So again, it is the system's internal time – the time defined by its evolution – that appears in the uncertainty relation for energy and time.

Problems

8.1 (a) Suppose we measure the electric field $\mathbf{E}$ inside a capacitor by sending a charged test particle through it. (See Fig. 8.2.) Assume first that the particle does not radiate. Show that the uncertainty in the measured value of the field strength is $\Delta p/eT$, where p is the transverse momentum of the particle, e its charge and T the time it takes to pass through the capacitor. (Neglect fringing of the electric field outside the capacitor.) Thus for large enough e, an accurate measurement of the electric field could be arbitrarily short.
(b) The uncertainty Δx in the transverse position of the particle cannot be greater than the separation of the the capacitor plates; the particle must not touch the plates. Show that $\Delta p \leq m \Delta x/T$ and that the transverse acceleration a of the particle satisfies $a \leq \Delta x/T^2$.
(c) According to Larmor's nonrelativistic formula [9], the rate at which a particle of charge e and acceleration a radiates energy is $2e^2a^2/3c^2$ (where c is the speed of light). Show that this rate implies a transverse momentum loss, over time T, of about $2e^2 \Delta x/3c^3T^2$. Since the momentum loss is uncertain, the uncertainty in the measure-

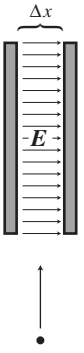

Figure 8.2: An experiment (Prob. 8.1) to measure the strength of the electric field **E** in a capacitor.

ment of the transverse electric field is not $\Delta p/eT$ but

$$\frac{1}{eT}\left[\Delta p + \frac{2}{3}\frac{e^2}{c^3}\frac{\Delta x}{T^2}\right] .$$

Show that the uncertainty is then greater than $\sqrt{\hbar c}/(cT)^2$, regardless of e, as in the Landau-Peierls paper [10]. (By contrast, Bohr and Rosenfeld [11] considered extended test bodies and did not use Larmor's formula.)

8.2 The neutral kaons (K-mesons) K_L and K_S have slightly different masses m_L and m_S and very different lifetimes τ_L and τ_S:

$$m_L/m_S - 1 \approx 10^{-14} , \quad \tau_L/\tau_S \approx 580 .$$

We can treat these mesons as mass eigenstates $|K_L\rangle$ and $|K_S\rangle$ that mix through strong and electromagnetic interactions.
(a) Show that the relative phase of any superposition of $|K_L\rangle$ and $|K_S\rangle$ is in principle measurable.
(b) Assume

$$|K_L\rangle = \frac{1}{\sqrt{2}}\left[|K^o\rangle - |\bar{K}^o\rangle\right] , \quad |K_S\rangle = \frac{1}{\sqrt{2}}\left[|K^o\rangle + |\bar{K}^o\rangle\right] .$$

Show that, without decay, the state $|K^o\rangle$ would evolve into the state $|\bar{K}^o\rangle$ in a time $T = \pi\hbar/(m_L - m_S)c^2 \approx 5.9 \times 10^{-10}$ sec. (But $\tau_S \approx 0.9 \times 10^{-10}$ sec.) As defined here, $|K_L\rangle$ and $|K_S\rangle$ are also eigenstates of CP. But experiments show that neutral kaon interactions are not quite invariant under CP, so $|K_L\rangle$ and $|K_S\rangle$ are not quite these linear combinations. It follows from the spin-statistics theorem [12] that the interactions are also not invariant under time reversal.

*8.3 Figure 8.3 shows an experiment [13] for measuring the speed of a charged particle. Initially, two heavy capacitors of length L move with speed V up the y-axis. The particle, of charge e, moves parallel to the x-axis with speed v_x (to be measured). The capacitors exert a force of magnitude eE on the particle, either to the right (inside the first capacitor) or to the left (inside the second capacitor).

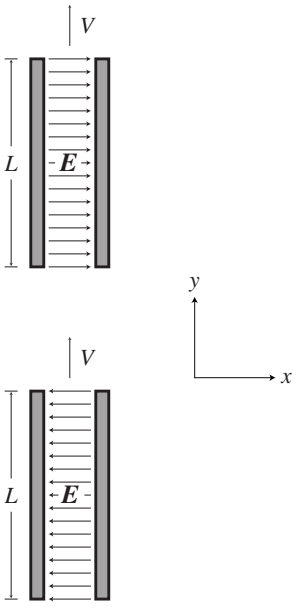

Figure 8.3: An experiment (Prob. 8.3) to measure the transverse speed of a charged particle.

(a) The momenta of the particle and capacitors change during the experiment. Use energy and momentum conservation to show that the forward momentum of each capacitor changes. But the force between the capacitors and the particle is, by assumption, parallel to the x-axis; how can the y-component of the momenta change? Resolve this paradox.
(b) Assume that the mass m of the particle is much smaller than the mass of either capacitor, that L is large (thus we can neglect the fringing of the electric field of the capacitors), and that the speed of the particle is always much smaller than V. Neglecting terms of higher order in v_x/V, etc., show that the initial speed v_x of the particle equals $KV/eEL - eEL/2mV$, where K is the change in kinetic energy of the first capacitor (initial minus final kinetic energy). Thus we can infer v_x from V, K, e, E, L and m.
(c) The first capacitor changes the transverse velocity of the particle. Show that the second capacitor restores the transverse velocity to its initial value v_x, and that the time of the measurement can be arbitrarily short.

8.4 Consider the Hamiltonian

$$H' = \frac{P^2}{2M} + \frac{p^2}{2m} + g(t)\frac{\mathbf{p}}{m} \cdot \mathbf{P} \ ,$$

which differs from H in Eq. (8.11) only in that $\mathbf{X}$ rather than $\mathbf{P}$ represents the pointer. Show that, in the limit of an impulsive measurement, the change in $\mathbf{X}$ measures $d\mathbf{x}/dt - g(t)\mathbf{P}/m$, which is a constant of the motion and equal to $\mathbf{v} = d\mathbf{x}/dt$ before and after

the measurement. Thus, in this limit, H' and H lead to equivalent measurements with compensation. Why are the measurements equivalent only in the limit of an impulsive measurement?

8.5 Argument iii) of Sect. 8.1 shows how to measure the energy of a particle by hitting it with a probe particle. Consider the interaction Hamiltonian

$$H_{int} = \frac{\mathbf{X}}{T} \cdot \mathbf{x}\,[\delta(t) - \delta(t-T)] - \frac{X^2}{2mT^2} \ ,$$

for a free particle of mass m. Here $\mathbf{x}$ is the coordinate of the free particle and $\mathbf{X}$ is the coordinate of a probe particle. Assume that $\mathbf{X}$ is nearly constant during the interval $0 \leq t \leq T$. Show that the momentum $\mathbf{P}$ conjugate to $\mathbf{X}$ yields a measurement of the velocity, and hence the energy, of the measured particle, and that the second interaction at time $t = T$ restores to the measured particle its original velocity. Thus we can evade the conclusion of this argument (Eq. (8.6) and its interpretation).

8.6 Consider a Hamiltonian to measure the average velocity $v = dx/dt$ of a particle over a time T:

$$H = \frac{P^2}{2M} + \frac{p^2}{2m} - \frac{xX}{T} + \frac{X^2}{12} \ .$$

Show, in the limit of large M, that $P(T) - P(0)$ corresponds to the average velocity $[x(T) - x(0)]/T$ that the particle *would* have if there were no measurement, not the average velocity that the particle *does* have during this measurement.

8.7 Consider the Hamiltonian

$$H = \frac{P^2}{2M} + \frac{p^2 + x^2}{2} - g(t)Xp - [g(t)X]^2 \frac{\sin T - T}{2T} \ ,$$

of a harmonically oscillating particle of unit mass and angular frequency. The coupling $g(t)$ is a constant g_0/T during the interval $0 \leq t \leq T$.
(a) In the approximation $M \gg g_0^2/T^2$, show that the change $P(T) - P(0)$ is proportional to what the average velocity dx/dt of the particle *would* have been during the time interval, if there had been no measurement.
(b) Show that the corresponding Lagrangian contains a harmonic oscillator potential for the coordinate X of the measuring device, with negative spring constant equal to $-[g(t)]^2(\sin T)/T$.

*8.8 (a) Quantize $\mathbf{A}(\mathbf{x})$ and $\mathbf{\Pi}(\mathbf{x})$ in a cubical box of side L with periodic boundary conditions, in the gauge $\nabla \cdot \mathbf{A} = 0$. Assume that no charges are present. Define the Fourier modes

$$\mathbf{A}(\mathbf{x}) = \frac{2c\sqrt{\pi}}{L^{3/2}} \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

and

$$\mathbf{\Pi}(\mathbf{x}) = \frac{1}{2c\sqrt{\pi}L^{3/2}} \sum_{\mathbf{k}} \mathbf{\Pi}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \quad + \quad \frac{1}{4\pi}\nabla V \ ,$$

where $\mathbf{A_k} = A_{\mathbf{k},1}\epsilon_1 + A_{\mathbf{k},2}\epsilon_2$, $\mathbf{\Pi_k} = \Pi_{\mathbf{k},1}\epsilon_1 + \Pi_{\mathbf{k},2}\epsilon_2$, and ϵ_1, ϵ_2 are unit vectors orthogonal to $\mathbf{k}$. Derive Eq. (8.12) starting from the density of energy in the electromagnetic field, $(E^2 + B^2)/8\pi$, and assuming $V = 0$.
(b) Starting from the Lagrangian density for the electromagnetic field, including interaction with a charge density $\rho(\mathbf{x})$ and current density $\mathbf{J}(\mathbf{x})$,

$$\begin{aligned}\mathcal{L}(\mathbf{x}) = & -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha \\ = & \frac{1}{8\pi}\left[\frac{1}{c^2}\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 - \frac{1}{2}(\nabla\times\mathbf{A})^2 + (\nabla V)^2 + \frac{2}{c}\nabla V\cdot\frac{\partial\mathbf{A}}{\partial t}\right] \\ & + \frac{1}{c}\mathbf{J}\cdot\mathbf{A} - \rho V\end{aligned}$$

(with indices α and β summed), derive the Hamiltonian density $\mathcal{H}(\mathbf{x})$:

$$\mathcal{H}(\mathbf{x}) = 2\pi c^2\mathbf{\Pi}^2 - c\nabla V\cdot\mathbf{\Pi} + \frac{1}{16\pi}(\nabla\times\mathbf{A})^2 - \frac{1}{c}\mathbf{J}\cdot\mathbf{A} + \rho V \ .$$

What are the degrees of freedom in $\mathcal{H}(\mathbf{x})$? Show, via the Lagrange equations of motion, that V is a function of ρ and not an independent degree of freedom. Show from this fact and from the gauge condition $\nabla\cdot\mathbf{A} = 0$ that the Fourier expansions in (a) contain all the degrees of freedom in $\mathbf{A}$ and $\mathbf{\Pi}$.
(c) Using the Fourier expansions in (a), show that $H = \int \mathcal{H}(\mathbf{x})d^3x$ is

$$H = H_{rad} - \frac{2\sqrt{\pi}}{L^{3/2}}\sum_{\mathbf{k}}\int \mathbf{A_k}\cdot\mathbf{J}(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}d^3x + \frac{1}{2}\int \rho V d^3x \tag{8.19}$$

if $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are localized. (For H_{rad} see Eq. (8.12).)

*8.9 In the Coulomb gauge, the scalar potential V is the instantaneous (i.e. not retarded) Coulomb potential of all charges. How can such a potential, which propagates instantaneously, be consistent [14] with relativistic causality?

8.10 The Fourier modes $\mathbf{A_k}$ in Eq. (8.12) and Prob. 8.8(a) are not all independent degrees of freedom. To eliminate the redundant degrees of freedom, we define $\mathbf{a_k} = a_{\mathbf{k},1}\epsilon_1 + a_{\mathbf{k},2}\epsilon_2$ where

$$a_{\mathbf{k},i} = \sqrt{\frac{\omega}{2\hbar}}A_{\mathbf{k},i} + \frac{i}{\sqrt{2\omega\hbar}}\Pi_{-\mathbf{k},i} \ .$$

(a) From the Fourier expansion of $\mathbf{A}(\mathbf{x})$ in Prob. 8.8(a) derive the constraint $\mathbf{A_{-k}} = \mathbf{A_k^\dagger}$ and show that

$$\mathbf{A_k} = \sqrt{\frac{2\omega}{\hbar}}\left(\mathbf{a_k} + \mathbf{a_{-k}^\dagger}\right)$$

automatically satisfies this constraint. Derive

$$\mathbf{A}(\mathbf{x}) = \frac{2c\sqrt{\pi\hbar}}{L^{3/2}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega}} \left[\mathbf{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \mathbf{a}_{\mathbf{k}}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$$

and

$$\mathbf{\Pi}(\mathbf{x}) = \frac{i\sqrt{\hbar}}{2c\sqrt{\pi}L^{3/2}} \sum_{\mathbf{k}} \sqrt{\frac{\omega}{2}} \left[\mathbf{a}_{\mathbf{k}}^{\dagger} e^{i\mathbf{k}\cdot\mathbf{x}} - \mathbf{a}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \right] \quad + \quad \frac{1}{4\pi} V \ .$$

Compare the expression for $\mathbf{\Pi}(\mathbf{x})$ with Eq. (3.2).
(b) Derive the commutation relations

$$[a_{\mathbf{k},i}, a_{\mathbf{k}',j}^{\dagger}] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{ij} \ .$$

(c) Show that H_{rad} in Eq. (8.12) formally equals

$$H_{rad} = \sum_{\mathbf{k}} \hbar\omega \left(\mathbf{a}_{\mathbf{k}}^{\dagger} \cdot \mathbf{a}_{\mathbf{k}} + \frac{1}{2} \right) \ ,$$

a sum of harmonic oscillators. The sum over all modes $\mathbf{k}$ of the ground state energy $\hbar\omega/2$ diverges, but a finite part of this divergence (the Casimir effect) has experimental significance [15].

8.11. Show that the commutator $[A_{\mathbf{k},i}, \Pi_{\mathbf{k}',j}] = i\hbar\delta_{ij}\delta_{\mathbf{k},\mathbf{k}'}$ (with $i, j = 1, 2, 3$) is incompatible with the constraint $\nabla \cdot \mathbf{E} = 0$ and with the gauge choice $\nabla \cdot \mathbf{A} = 0$. Show that the commutator $[A_{\mathbf{k},i}, \Pi_{\mathbf{k}',j}] = i\hbar(\delta_{ij} - k_i k_j/k^2)\delta_{\mathbf{k},\mathbf{k}'}$ is compatible with $\nabla \cdot \mathbf{E} = 0$ and with $\nabla \cdot \mathbf{A} = 0$ and that, in the gauge $\nabla \cdot \mathbf{A} = 0$, the two commutators lead to equivalent equations of motion.

8.12. (a) Derive Eq. (8.14).
(b) Show that the equation of motion for $\mathbf{A}_{\mathbf{k}}$ can be written

$$\frac{dA_{\mathbf{k},x}}{dt} = \Pi_{-\mathbf{k},x} - g(t)X\rho_{\mathbf{k}}(1 - k_x^2/k^2) \ ,$$
$$\frac{dA_{\mathbf{k},y}}{dt} = \Pi_{-\mathbf{k},y} + g(t)X\rho_{\mathbf{k}}k_x k_y/k^2 \ ,$$
$$\frac{dA_{\mathbf{k},z}}{dt} = \Pi_{-\mathbf{k},z} + g(t)X\rho_{\mathbf{k}}k_x k_z/k^2 \ .$$

*8.13 Derive Eq. (8.16). (Note Prob. 8.7.)

*8.14 Define the vacuum of the electromagnetic field to be the state $|0\rangle$ such that $a_{\mathbf{k},i}|0\rangle = 0$ for all $\mathbf{k}$ and i. (Prob. 8.10 defines $a_{\mathbf{k},i}$, and $|0\rangle$ is the ground state of all the harmonic operators in Prob. 8.10(c).) Define a normalized distribution

$$\rho_D(\mathbf{x}) = \frac{e^{-|\mathbf{x}|^2/D^2}}{\pi^{3/2}D^3}$$

and an operator $\mathbf{E}_D = \int d^3x \mathbf{E}(\mathbf{x})\rho_D(\mathbf{x})$. Prove that

$$\langle 0|\mathbf{E}_D^2|0\rangle = \frac{4\hbar c}{\pi D^4} \ .$$

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9 Quantum Cats

Following von Neumann, we have defined a general quantum measurement by treating the measuring device *as a quantum system*. Quantum measurements are an essential tool for investigating quantum mechanics, but they lead to a paradox. We have avoided this paradox by discussing quantum measurements in the Heisenberg formalism. Heisenberg's equations of motion are convenient because they are identical to the corresponding classical equations of motion, but the identity is only formal and obscures the paradox. In this chapter, we look at quantum measurements in the Schrödinger formalism, and the paradox hits us over the head.

The paradox is not a mere artifact of the von Neumann model. It arises from the division of the world into quantum and classical parts, with measuring devices residing in the classical part. But even if the measuring devices reside in the quantum part – as in quantum measurements – the division remains, for observers reside in the classical part; observations are actual, not just possible. The division of the world into classical and quantum parts is paradoxical. A revolution replaces an old system with a new one; but in quantum mechanics, the new system depends on the old one, classical mechanics. Quantum mechanics is an unfinished revolution.

9.1 Schrödinger's Cat

The paradox can arise in any quantum measurement. For example, consider an atom with spin $\hbar/2$. An interaction Hamiltonian for measuring the z-component of the spin of the atom, $S_z = \hbar\sigma_z/2$, is

$$H_{int}(t) = g(t)P_d S_z \ . \tag{9.1}$$

The momentum P_d is conjugate to the coordinate Q_d of the measuring device. The measuring device could be a Stern-Gerlach apparatus, as in Sect. 7.2; then Q_d would correspond to the transverse deflection of the atom. However, we are not concerned here with the apparatus; we simply identify Q_d with the position of a pointer on a measuring device. As in Eq. (7.4), $g(t)$ is nonzero only for $0 \le t \le T$ and satisfies

$$\int_0^T dt\ g(t) = g_0 \ .$$

We assume that the measurement is impulsive ($T \to 0$) so that we can regard H_{int} as the total Hamiltonian during the measurement. Then in the Heisenberg formalism, $Q_d(T) - Q_d(0) = \int_0^T dt\ \dot{Q}_d(t) = g_0 S_z$.

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In the Schrödinger formalism, quantum states evolve in time. Suppose the initial spin state of the atom is $|\uparrow\rangle$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ represent spin up and down the z-axis, respectively. Let the initial state of the measuring device be sharply peaked around $Q_d = 0$, and denote it $|0\rangle$. The combined initial state is

$$|\Psi(0)\rangle = |\uparrow, 0\rangle \equiv |\uparrow\rangle \otimes |0\rangle \ .$$

The time evolution operator during the measurement (when H_{int} dominates the rest of the Hamiltonian) is

$$U(T,0) = e^{-i\int_0^T dt\ H_{int}(t)/\hbar} = e^{-ig_0 P_d \sigma_z/2} \ , \tag{9.2}$$

When $U(T,0)$ acts on the initial state $|\Psi(0)\rangle$, the operator σ_z reduces to 1 because $|\Psi(0)\rangle$ is an eigenstate of σ_z with eigenvalue 1. Then

$$\begin{aligned} |\Psi(T)\rangle &= U(T,0)\,|\Psi(0)\rangle \\ &= e^{-ig_0 P_d \sigma_z/2}\,|\uparrow, 0\rangle \\ &= e^{-ig_0 P_d/2}\,|\uparrow, 0\rangle \\ &= |\uparrow, g_0\hbar/2\rangle \ , \end{aligned}$$

i.e. the pointer is displaced by $g_0\hbar/2$. (See Eq. (5.7).) Likewise, if the initial spin state of the atom is $|\downarrow\rangle$, the combined initial state is $|\downarrow, 0\rangle \equiv |\downarrow\rangle \otimes |0\rangle$ and evolves during the measurement to the state $|\downarrow, -g_0\hbar/2\rangle$. The spin S_z does not change, while the pointer on the measuring device shifts in proportion to S_z, whatever it is – just as in the Heisenberg formalism.

Now what if the initial spin state of the atom is neither $|\uparrow\rangle$ nor $|\downarrow\rangle$ but a superposition of the two? For example, the initial spin state of an atom polarized along the x-axis is $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. Let it interact with a measuring device in the state $|0\rangle$ as before; the combined initial state is

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle + |\downarrow\rangle\right] \otimes |0\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow, 0\rangle + |\downarrow, 0\rangle\right] \ . \tag{9.3}$$

The time evolution operator $U(T,0)$ is linear, so the state at the end of the measurement is

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}\left[\,|\uparrow, g_0\hbar/2\rangle \ + \ |\downarrow, -g_0\hbar/2\rangle\,\right] \ . \tag{9.4}$$

What does this state represent? It represents nothing that we have ever encountered: a measuring device in a superposition of two orthogonal states, corresponding to two different values of S_z. We are used to superpositions of quantum systems, but not of pointers – we expect the pointer of a measuring device to be in a definite position.

Clearly, our treatment of quantum measurements is incomplete; we cannot leave the measuring device in a superposition of states. But clearly, quantum mechanics offers no way to reduce a superposition of pointer positions to a definite position. The Schrödinger equation implies that the time evolution operator $U(T,0)$ is unitary and linear. (See Sect. 5.5.) Since it is linear, it preserves superpositions, as we just saw. Since it is unitary, it preserves the scalar product between any two states. If $U(T,0)$ could evolve the initial state $|\Psi(0)\rangle$ in Eq. (9.3)

sometimes to $|\Psi(T)\rangle = |\uparrow, g_0\hbar/2\rangle$ and sometimes to $|\Psi(T)\rangle = |\downarrow, -g_0\hbar/2\rangle$, then $U(T,0)$ could evolve the same initial state $|\Psi(0)\rangle$ to two final states that are orthogonal to each other. Unitarity forbids such evolution.

Unitary evolution cannot reduce the superposition of pointer positions, so von Neumann postulated a second kind of time evolution: when a measurement is complete, the superposition of positions collapses to a definite position. According to this "collapse postulate", the superposition $|\Psi(T)\rangle$ in Eq. (9.4) then collapses into one of the two product states, $|\uparrow, g_0\hbar/2\rangle$ or $|\downarrow, -g_0\hbar/2\rangle$, with equal probability. The collapse postulate states that if a measurement leaves the measuring device and measured system in a superposition of product states, the superposition automatically collapses. The probability of each product state equals the absolute value squared of its coefficient in the superposition. Thus quantum measurements confirm the statistical predictions of quantum mechanics.

But when is a measurement complete? What is a measuring device? The collapse postulate leaves these questions open. For example, could the measuring device be another spin? Let us apply H_{int} in Eq. (9.1) as follows: alongside $S_z = \sigma_z/2$ which represents the atomic spin, we define σ_y^d and σ_z^d as Pauli spin matrices for the "measuring device". We set $P_d = \sigma_y^d$ and $g_0 = \pi/2$; the initial state of the "measuring device" will be $|0\rangle_d = (|\uparrow\rangle_d + |\downarrow\rangle_d)/\sqrt{2}$. The time evolution operator $U(T,0)$ in Eq. (9.2) is now

$$U(T,0) = e^{-i\pi\sigma_z\sigma_y^d/4} , \tag{9.5}$$

and rotates the "measuring device" in the xz-plane by $\pm\pi/2$. If the atomic spin state is $|\uparrow\rangle$, then $U(T,0)$ is

$$U(T,0) = e^{-i\pi\sigma_y^d/4} ,$$

and the final state of the two spins is $|\uparrow,\downarrow\rangle \equiv |\uparrow\rangle \otimes |\downarrow\rangle_d$. If the atomic spin state is $|\downarrow\rangle$, the final state is $|\downarrow,\uparrow\rangle \equiv |\downarrow\rangle \otimes |\uparrow\rangle_d$; and if the initial atomic spin state is $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, the final state of the two spins is

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle\right) . \tag{9.6}$$

The state $|\Psi(T)\rangle$ in Eq. (9.6) is entangled. We can experimentally verify that the collapse postulate does *not* apply to $|\Psi(T)\rangle$. (See Prob. 9.1.) The collapse postulate does not apply to this "measuring device", but it conceivably applies to more complicated systems. How complicated? Where is the dividing line between unitary time evolution and collapse? The same questions arise in regard to Bohr's description of the measurement process.

Schrödinger dramatized the question with a thought experiment [1]: "A cat is placed in a steel chamber, together with the following hellish contraption (which must be protected against direct interference by the cat): In a Geiger counter there is a tiny amount of radioactive substance, so tiny that maybe within an hour one of the atoms decays, but equally probably none of them decays. If one decays then the counter triggers and via a relay activates a little hammer which breaks a container of cyanide." (See Fig. 9.1.) So after an hour there is an equal chance of finding the cat alive or dead. Schrödinger's initial state $|\Psi(0)\rangle$,

$$|\Psi(0)\rangle = |\text{undecayed}\rangle \otimes |\text{untriggered}\rangle \otimes |\text{unactivated}\rangle \otimes |\text{unbroken}\rangle \otimes |\text{live}\rangle , \tag{9.7}$$

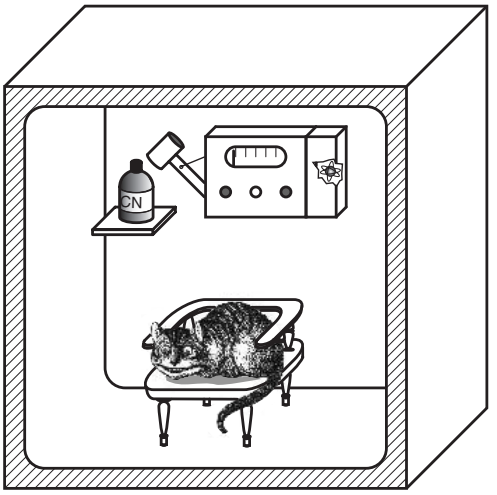

Figure 9.1: Schrödinger's "hellish contraption", including radioactive atoms, a Geiger counter, a hammer, a bottle of cyanide, and a cat blissfully unaware of it all.

represents the atoms, the Geiger counter, the hammer, the container of cyanide, and the cat, respectively. After a time T equal to an hour, unitary evolution transforms $|\Psi(0)\rangle$ into

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}|\text{undecayed}\rangle \otimes |\text{untriggered}\rangle \otimes |\text{unactivated}\rangle \otimes |\text{unbroken}\rangle \otimes |\text{live}\rangle + \frac{1}{\sqrt{2}}|\text{decayed}\rangle \otimes |\text{triggered}\rangle \otimes |\text{activated}\rangle \otimes |\text{broken}\rangle \otimes |\text{dead}\rangle \ . \tag{9.8}$$

Does the collapse postulate apply to $|\Psi(T)\rangle$ or doesn't it? This question has moral consequences (if the cat dies). If $|\Psi(T)\rangle$ collapses before we look into the steel chamber, then whoever set up the "hellish contraption" is the murderer. But if the state remains $|\Psi(T)\rangle$ up to the time we look into the chamber, we are the murderers – the cat was in a superposition until we looked.

9.2 A Quantum Catalyst

Einstein, Podolsky and Rosen claimed that quantum mechanics is incomplete. Their claim may be correct – in a way they did not anticipate, as Sect. 3.5 notes. Quantum mechanics is incomplete because it does not account for the actual results of measurements. As the paradox of Schrödinger's cat shows, unitary evolution cannot turn possible results into actual results. Aware of this paradox, von Neumann postulated collapse. But von Neumann's collapse is at best an effective model; it does not resolve the paradox.

Attempts to resolve the paradox fall into three classes, corresponding to three statements:

i) Quantum mechanics is incomplete and there is collapse.

ii) Quantum mechanics is incomplete and there is no collapse.

iii) Quantum mechanics is complete.

Explicit proposals for von Neumann's collapse are consistent with statement i). A proposal consistent with statement ii) is Bohm's theory of hidden variables. (See Sect. 3.5 and Prob. 9.3.)

In Bohm's theory there is no need for collapse, because hidden variables account for the results of all measurements. Section 9.3 presents a surprising proposal consistent with statement iii), namely the Everett–Wheeler interpretation of quantum mechanics. How can quantum mechanics be complete, when unitary evolution cannot turn possible results into actual results? The answer to this question is surprising indeed.

Proposals to resolve the paradox according to statement i) are radical, in that they all modify quantum mechanics. They all modify unitary time evolution by postulating some form of collapse. Here we consider three such proposals.

The first proposal is due to Wigner, who noted that we never find ourselves in a superposition of states. If we look at the pointer on a measuring device, we always find the pointer in a definite position. But according to Eq. (9.4), a pointer may be in a superposition of two positions. According to Eq. (9.4), if we look at the pointer after a measurement, we too should evolve into a superposition of two states, one state for each definite position of the pointer. But we are not in a superposition! About cats we may have doubts, but not about ourselves. Since Eq. (9.4) holds for unitary evolution, our evolution cannot be unitary. Wigner [2] suggested that collapse occurs whenever a conscious human being observes a measuring device in a superposed state. But this suggestion reduces something that is not well defined (collapse) to something else that is not well defined (consciousness). The phenomena of science are public while consciousness is private.

Another proposal, due to Penrose [3], starts with the assumption that there is no way to quantize gravity without modifying quantum mechanics. Penrose proposed that the superposed states of any system (not just of measuring devices) collapse under the influence of gravitational fields. Weak gravitational fields, such as the gravitational field of a molecule, do not interfere with unitary evolution; but strong gravitational fields modify the evolution by inducing collapse. The transition from "weak" to "strong" gravity should occur at a scale of about one Planck mass;[1] systems much more massive than the Planck mass would behave classically. So far this suggestion, too, is not well defined, since quantum gravity is not well defined. Diosi, however, made a similar and concrete suggestion [4].

The most concrete proposals for collapse are the "Spontaneous Localization" (SL) model of Ghirardi, Rimini and Weber and the "Continuous Spontaneous Localization" (CSL) model of Pearle [5]. In both models, extended states of matter spontaneously collapse to localized states of size $a \approx 10^{-5}$ cm. In the SL model, an extended state $\psi(\mathbf{x})$ can spontaneously localize to $\psi(\mathbf{x})g(\mathbf{x}-\bar{\mathbf{x}})$, where $g(\mathbf{x}-\bar{\mathbf{x}})$ is a gaussian of width a centered at $\mathbf{x}=\bar{\mathbf{x}}$. The probability of such a localization is proportional to

$$\int d^3x |\psi(\mathbf{x})g(\mathbf{x}-\bar{\mathbf{x}})|^2 \ .$$

Thus the SL model is consistent with Born's statistical interpretation of the wave function. But what happens to quantum interference if $\psi(\mathbf{x})$ spontaneously localizes? The answer of the SL model is that $\psi(\mathbf{x})$ spontaneously collapses, but each second the probability of a "hit" – a collapse – is about $\lambda = 10^{-16}$, i.e. it happens once in about 300 million years on average. For all practical purposes, the wave function of a given particle *never* collapses, and so we see such

[1] The Planck mass, defined to be $\sqrt{\hbar c/G} \approx 2.18 \times 10^{-5}$ g (where G is Newton's gravitational constant), is about the mass of a baby tick, and about 10^{17} times the mass of a heavy atom.

quantum interference effects as electron diffraction. But a *macroscopic* object, such as the pointer on a measuring device, contains many particles. Imagine a pointer in a superposition of two positions separated by a distance $L \gg a$. We might represent the initial state $|\Psi(0)\rangle$ as

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |L\rangle\right] ,$$

where $|0\rangle$ and $|L\rangle$ denote wave functions of the pointer position that are sharply peaked at 0 and at L, respectively. But the pointer contains some 10^{22} particles, each with degrees of freedom of its own. Let i index the particles and let $|\psi_0^{(i)}\rangle$ and $|\psi_L^{(i)}\rangle$ represent the complete wave function of the i-th particle in the pointer when the pointer position is 0 and L, respectively; assume $|\psi_0^{(i)}\rangle$ and $|\psi_L^{(i)}\rangle$ do not overlap. Let us take the terms $|0\rangle$ and $|L\rangle$ to be tensor products of these one-particle wave functions:[2]

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left[\bigotimes_i |\psi_0^{(i)}\rangle + \bigotimes_i |\psi_L^{(i)}\rangle\right] \tag{9.9}$$

The index i runs over N values, with N of order 10^{22}. Suppose that just one of the particles, say the n-th, spontaneously localizes. That is, its wave function collapses to a new wave function localized in a region of size $a \ll L$. The new wave function for the n-th particle could overlap with either $|\psi_0^{(n)}\rangle$ or $|\psi_L^{(n)}\rangle$ but not both. Thus either $|\psi_0^{(n)}\rangle$ or $|\psi_L^{(n)}\rangle$ would vanish – and along with it, the corresponding tensor product term in Eq. (9.9). The rate for this spontaneous localization would be about $10^{22} \times 10^{-16}\ \mathrm{sec}^{-1} = 10^6\ \mathrm{sec}^{-1}$, so the pointer would have a definite position after approximately 10^{-6} seconds; the collapse of any macroscopic superposition would be extremely rapid.

In the CSL model, the time evolution is continuous rather than sudden. An operator $\rho(\mathbf{x}, t)$ represents the number of particles in a sphere of radius $a \approx 10^{-5}$ cm, centered at $\mathbf{x}$ at time t. (The operator $\rho(\mathbf{x}, t)$ could represent the mass, rather than the number, of the particles in the sphere; if so the CSL model, like Diosi's model, is a concrete form of Penrose's suggestion.) The evolution of a state vector depends on $\rho(\mathbf{x}, t)$ and also on a classical field $w(\mathbf{x}, t)$ with a measure, i.e. a probability density for each spacetime configuration of $w(\mathbf{x}, t)$. (See Prob. 9.4.) Since the evolution depends on $w(\mathbf{x}, t)$, we let $|\Psi_w(t)\rangle$ rather than $|\Psi(t)\rangle$ denote the state vector for $t > 0$. The evolution equation for $|\Psi_w(t)\rangle$ (which by definition is a multiparticle wave function) is

$$\frac{\partial}{\partial t}|\Psi_w(t)\rangle = -\frac{i}{\hbar}H|\Psi_w(t)\rangle - \frac{1}{4a^3\lambda}\int d\mathbf{x}\left[w(\mathbf{x}, t) - 2\lambda\rho(\mathbf{x}, t)\right]^2 |\Psi_w(t)\rangle , \tag{9.10}$$

where H is the Hamiltonian for the particle. Equation (9.10) assigns a unique nonunitary evolution $|\Psi_w(t)\rangle$ to each configuration of $w(\mathbf{x}, t)$. (It is nonunitary because the norm of $|\Psi_w(t)\rangle$ may change.) Now the probability density that the initial state vector actually evolves according to $|\Psi_w(t)\rangle$ during an interval $0 \le t \le T$ is equal to $\langle\Psi_w(T)|\Psi_w(T)\rangle$ times the probability density for the configuration of $w(\mathbf{x}, t)$.

[2] There could be entanglement within $|0\rangle$ and $|L\rangle$, but for simplicity (to keep the sum to two terms) we take the terms to be tensor products.

As a simple application of the CSL model, let us take $H = 0$ and consider a measuring device with a pointer consisting of a steel needle, 1 cm long, with cross section 2×10^{-3} cm^2. The initial state is $|\Psi(0)\rangle$ of Eq. (9.9); as before, $|\psi_0^{(i)}\rangle$ and $|\psi_L^{(i)}\rangle$ represent the complete wave function of the i-th particle in the pointer when the pointer is displaced a distance 0 and L, respectively. Without representing the operator $\rho(\mathbf{x}, t)$ explicitly, we assume that it acts on each term in the superposition to yield N at any point on the pointer and zero elsewhere, where N is the number of particles in a steel ball of radius $a = 10^{-5}$ cm. Since the coefficient of the integral in Eq. (9.10) is negative, the norm of $|\Psi_w(t)\rangle$ tends to decrease over time; it remains unchanged only if $w(\mathbf{x}, t)$ precisely cancels $2\lambda\rho(\mathbf{x}, t)$. Thus of all possible configurations of $w(\mathbf{x}, t)$, two classes are important for the evolution of $|\Psi(0)\rangle$. One has $w(\mathbf{x}, t)$ on average equal to $2\lambda N$ at the position of the undisplaced pointer, and zero elsewhere. The other has $w(\mathbf{x}, t)$ on average equal to $2\lambda N$ at the position of the pointer displaced by L, and zero elsewhere. In the first case $|\Psi(0)\rangle$ evolves to the state $|0\rangle = \bigotimes_i |\psi_0^{(i)}\rangle$, and in the second case it evolves to the state $|L\rangle = \bigotimes_i |\psi_L^{(i)}\rangle$; the collapse takes a time T of order 6×10^{-18} sec. (See Prob. 9.4.)

The CSL model is (so far) consistent with experiment. Unlike the SL model, it preserves the symmetry or antisymmetry of multiparticle states, because the operator $\rho(\mathbf{x}, t)$ commutes with particle exchange. Pearle has proposed relativistic CSL models for free particles [6]. A defect of these models, so far, is that the random classical field induces production of particles out of the vacuum.

9.3 Quantum Concatenations

The proposals in the previous section, as well as Bohm's theory, all assume that quantum mechanics is incomplete. In 1957 Everett, a student of Wheeler, challenged this assumption. Everett interpreted quantum mechanics as a theory without collapse. He called his interpretation the "relative states" formulation of quantum mechanics; it is also known as the Everett-Wheeler or Many Worlds interpretation of quantum mechanics [7]. Everett and Wheeler proposed that quantum mechanics, with unitary time evolution, is a complete theory. In this respect only, the proposal of Everett and Wheeler is conservative where the proposals of the last section are radical. Quantum mechanics is complete – if we can believe it!

How do Everett and Wheeler resolve the paradox of Schrödinger's cat? Recall the chain of events. First, an atom may or may not decay. All the other events in the chain involve macroscopic systems: a Geiger counter, a hammer, a container of cyanide, a cat. The state of the "hellish contraption" after a hour of time evolution is given by Eq. (9.8). Equation (9.8) does not tell us whether the cat is dead or alive; it merely offers two possible accounts. If an atom decayed, all the events involving the macroscopic systems occurred: the Geiger counter clicked and activated the hammer and the hammer broke the bottle of cyanide, killing the cat. If no atom decayed, none of the other events took place. Each state represents a completely self-consistent account. According to Everett and Wheeler, *both these accounts are real.* In a single world, only one account can be real. But in many worlds, they can both be real. Many accounts imply many worlds, hence the name "Many Worlds interpretation".

To formalize this proposal, let us define $|s_i\rangle$ to be a set of normalized states of a system S and $|d_j\rangle$ a set of normalized states of a system D. Suppose the combined state of the two

systems is

$$\sum_{ij} c_{ij}|s_i, d_j\rangle \ .$$

In general, the states of S and D are correlated and we cannot define the state of D alone. The most we can do is choose a state $|s_n\rangle$ of the system S and define the *relative state* of D (relative to this state of S) to be

$$N \sum_j c_{nj}|d_j\rangle \ ,$$

where N is a constant that normalizes the state. The relative state of the system D is simply the state of D if we project the combined state of S and D onto $|s_n\rangle$ and normalize. Similarly, we can define the state of S *relative to* a state of D.

We are interested in measurements, so let D be a measuring device acting on the system S. Let the states $|s_i\rangle$ be eigenstates of some observable and let the measuring device D measure that observable. It is convenient to let $|d_i\rangle$ be the state of the measuring device indicating that the system S is in the state $|s_i\rangle$; that is, at the end of a measurement on S in the state $|s_i\rangle$, D is in the state $|d_i\rangle$. Let the initial state of the measuring device be $|d_0\rangle$; the evolution

$$|s_i, d_0\rangle \to |s_i, d_i\rangle \ ,$$

for any i, defines the measurement. In general, the initial state of the system S is not one of the states $|s_i\rangle$ but a superposition $\sum_i c_i|s_i\rangle$. If the initial combined state is $\sum_i c_i|s_i, d_0\rangle$, unitary time evolution during the measurement takes this initial state to

$$\sum_i c_i|s_i, d_0\rangle \to \sum_i c_i|s_i, d_i\rangle \ .$$

The final state is a superposition and we cannot define a state for either S or D alone; at most we can say that if S is in the state $|s_i\rangle$, then D is in the relative state $|d_i\rangle$, and vice versa.

We can generalize this notation and let $D^{(1)}, D^{(2)}, D^{(3)}, \ldots$ represent identical measuring devices (or observers) and $S^{(1)}, S^{(2)}, S^{(3)}, \ldots$ represent identical systems on which they may act. All the measuring devices measure an operator which has normalized eigenstates $|s_i^{(1)}\rangle$ on system $S^{(1)}$, normalized eigenstates $|s_i^{(2)}\rangle$ on system $S^{(2)}$, and so on. The initial state of $D^{(1)}$ is $|d_0^{(1)}\rangle$, the initial state of $D^{(2)}$ is $|d_0^{(2)}\rangle$, and so on. For any system $S^{(m)}$ in an initial state $|s_i^{(m)}\rangle$, and for any measuring device $D^{(n)}$, a measurement corresponds to the evolution

$$|s_i^{(m)}, d_0^{(n)}\rangle \to |s_i^{(m)}, d_i^{(n)}\rangle \ .$$

Now consider a possible measurement scenario:

i) Two measuring devices, $D^{(1)}$ and $D^{(2)}$, measure the same observable on a system S. $D^{(1)}$ measures first, then $D^{(2)}$. The evolution can be described by the sequence

$$\sum_i c_i|s_i, d_0^{(1)}, d_0^{(2)}\rangle \to \sum_i c_i|s_i, d_i^{(1)}, d_0^{(2)}\rangle \to \sum_i c_i|s_i, d_i^{(1)}, d_i^{(2)}\rangle \ . \tag{9.11}$$

Each value of i in the sums of Eq. (9.11) corresponds to a different state of the system S. Relative to a state $|s_i\rangle$ of S, the combined state of the measuring devices is $|d_i^{(1)}, d_i^{(2)}\rangle$, so the two measuring devices always agree. Any number of measuring devices measuring the same observable on the system S would always agree, relative to the state $|s_i\rangle$, that S is in the state $|s_i\rangle$. Relative to the state $|s_i\rangle$, measuring devices agree on an account – for each i! According to the Many Worlds interpretation, each of these accounts is real.

The Many Worlds interpretation boggles the mind, but could it be right? Does it yield quantum probabilities? Consider measurements on a system S in the initial state $c_1|s_1\rangle + c_2|s_2\rangle$ with $|c_1|^2 \neq |c_2|^2$. They yield two different accounts, S in the state $|s_1\rangle$ and S in the state $|s_2\rangle$. If each of these accounts corresponds to one world, then what do the coefficients c_1 and c_2 have to do with probabilities? Why are we more likely to find S in one state than in the other? To answer these questions, we consider two more measurement scenarios:

ii) Let a measuring device D act on identical systems $S^{(1)}$ and $S^{(2)}$, prepared in the same initial states $\sum_i c_i|s_i^{(1)}\rangle$ and $\sum_j c_j|s_j^{(2)}\rangle$, respectively. Extending our notation slightly, we denote by $|d_{ij}\rangle$ a state of the measuring device that indicates that the system $S^{(1)}$ is in the state $|s_i^{(1)}\rangle$ and the system $S^{(2)}$ is in the state $|s_j^{(2)}\rangle$. The initial state of the measuring device is still $|d_0\rangle$. During the measurement the evolution is

$$\sum_{ij} c_i c_j |s_i^{(1)}, s_j^{(2)}, d_0\rangle \to \sum_{ij} c_i c_j |s_i^{(1)}, s_j^{(2)}, d_{ij}\rangle \ .$$

(iii) Let a measuring device D measure the same observable on many identical systems $S^{(j)}$, prepared in the same initial state $\sum_i c_i|s_i^{(j)}\rangle$. During the measurement, the evolution is

$$\sum_{ij\ldots} c_i c_j \ldots |s_i^{(1)}, s_j^{(2)}, \ldots, d_0\rangle \to \sum_{ij\ldots} c_i c_j \ldots |s_i^{(1)}, s_j^{(2)}, \ldots, d_{ij\ldots}\rangle \ , \tag{9.12}$$

where $|d_{ij\ldots}\rangle$ extends the notation $|d_{ij}\rangle$ to measurements on an arbitrary number of identical systems. The right side of Eq. (9.12) is a superposition of terms representing all the systems $S^{(1)}, S^{(2)}, \ldots$ in definite states with the measuring device indicating those states. Thus each term corresponds to a definite account of the measurement and each account is incompatible with all the other accounts.[3] Furthermore, each term in the sum in Eq. (9.12) appears with a coefficient. The term that represents system $S^{(1)}$ in the state $|s_i^{(1)}\rangle$, system $S^{(2)}$ in the state $|s_j^{(2)}\rangle$, system $S^{(3)}$ in the state $|s_k^{(3)}\rangle$, and so on, appears with coefficient $c_i c_j c_k \ldots$. The square of the absolute value of this coefficient is

$$|c_i c_j c_k \ldots|^2 = |c_i|^2 \cdot |c_j|^2 \cdot |c_k|^2 \cdot \ldots \ . \tag{9.13}$$

In the conventional interpretation of quantum mechanics, Eq. (9.13) represents the probability that system $S^{(1)}$ is in the state $|s_i^{(1)}\rangle$, system $S^{(2)}$ is in the state $|s_j^{(2)}\rangle$, system $S^{(3)}$ is in the state $|s_k^{(3)}\rangle$, and so on. What does Eq. (9.13) represent if there is no collapse?

[3] There could be additional measuring devices besides D in this scenario. However, we have already seen in scenario i) that all measuring devices agree, relative to states chosen for all the systems. So we can let D stand for all measuring devices acting on all the systems to measure the same observable.

Everett interpreted Eq. (9.13) as a *measure* of the account that has $S^{(1)}$ in the state $|s_i^{(1)}\rangle$, $S^{(2)}$ in the state $|s_j^{(2)}\rangle$, $S^{(3)}$ in the state $|s_k^{(3)}\rangle$, and so on. That is, the correspondence between worlds and accounts is not one-to-one but many-to-one. To paraphrase Orwell [8], all accounts are real, but some are more real than others. The measure is what maps coefficients c_1 and c_2 in an initial state $S = c_1|s_1\rangle + c_2|s_2\rangle$ to probabilities $|c_1|^2$ and $|c_2|^2$ of final states $|s_1\rangle$ and $|s_2\rangle$, respectively.

For example, suppose $c_1 = \sqrt{1/3}$ and $c_2 = \sqrt{2/3}$. The measure implies that for every world in which a measurement yields the final state $|s_1\rangle$, there are two worlds in which a measurement yields the final state $|s_2\rangle$. Let us choose a world at random. How do we choose a world at random? We measure the final state. From the result of our measurement, we know if we are in the world with final state $|s_1\rangle$ or in one of the two worlds with final state $|s_2\rangle$. What is the probability that the final state is $|s_2\rangle$ in the world we choose? The measure insures that it is 2/3, the quantum probability of the final state $|s_2\rangle$. In this example, the probabilities are rational; but even if they are not rational, the measure maps coefficients c_i to probabilities $|c_i|^2$.

The mapping arises as follows. Suppose that the probabilities of events $E_1, E_2, E_3, \ldots$ are $p_1, p_2, p_3, \ldots$ respectively. Then the probability of a sequence of uncorrelated events $E_i, E_j, E_k, \ldots$ is $p_i \cdot p_j \cdot p_k \cdot \ldots$. The converse is also true: if the probability of a sequence of uncorrelated events $E_i, E_j, E_k, \ldots$ is $p_i \cdot p_j \cdot p_k \cdot \ldots$, then the probabilities of events $E_1, E_2, E_3, \ldots$ are $p_1, p_2, p_3, \ldots$ respectively. In scenario (iii) the sequence of events $i, j, k, \ldots$ (i.e. the sequence of measurements leaving $S^{(1)}$ in the state $|s_i^{(1)}\rangle$, $S^{(2)}$ in the state $|s_j^{(2)}\rangle$, $S^{(3)}$ in the state $|s_k^{(3)}\rangle$, etc.) has measure $|c_i|^2 \cdot |c_j|^2 \cdot |c_k|^2 \cdot \ldots$. If we choose a world at random (by making these measurements), the probability of this sequence equals its measure, $|c_i|^2 \cdot |c_j|^2 \cdot |c_k|^2 \cdot \ldots$. Therefore, the probability that a measurement on a system $S^{(m)}$ will yield $|s_i^{(m)}\rangle$ *in that world* is $|c_i|^2$, just as quantum mechanics predicts.

Thus in the Many Worlds interpretation, quantum mechanics – with unitary time evolution – is a complete theory. Quantum mechanics is causal and deterministic, yet it implies the same experimental predictions as other interpretations of quantum mechanics. Only experimental predictions *about collapse itself* would not be the same. If there could be an experiment to test for collapse, the experiment would test the Many Worlds interpretation. It would also test Bohm's theory of hidden variables, in which there is no collapse. (See Sect. 3.4.) But it would not distinguish between the Many Worlds interpretation and Bohm's theory.

9.4 A Quantum Catalog

This chapter opens with a paradox involving a quantum cat, Schrödinger's cat. The resolution of this paradox seems to involve more quantum cats. According to Wigner, consciousness resolves the paradox, so his resolution is a "quantum catharsis". Penrose's resolution is a "quantum catapult" – he throws a large mass at the paradox. The CSL model contains a "quantum catalyst" – a random field that induces collapse but does not itself change. Bohm's theory contains "quantum catacombs", hidden connections across space and time. And in the Many Worlds interpretation, "quantum concatenations" – unending chains of measurements – define the worlds.

All these cats are alive insofar as nothing in experiment or theory kills them; we have not proved or disproved any of these resolutions.[4] Hence the resolution of the paradox remains a matter of taste. The CSL model is testable, but so far there is no evidence for collapse. To falsify collapse, on the other hand, we must verify that no superposition ever collapses. For example, we must show that Schrödinger's cat remains in an entangled state – and in practice, we have no hope of showing that the state remains entangled. The entangled state is a sum of two terms, corresponding to the live and dead cat. In Eq. (9.8), each term appears as a tensor product of five states. But actually each is a tensor product of countless states representing every observable inside the steel chamber, such as the position of every atom on the tip of the cat's tail. If we lose track of some observables – e.g. if even a few molecules of gas leak out of the steel chamber – we have lost track of the relative phase between terms, and we cannot verify the entangled state of Schrödinger's cat. (See Prob. 9.7.)

Hence *in practice*, we cannot verify the entangled state of Schrödinger's cat. Can we verify the state *in principle*? If so, in what sense? In principle, any quantum operator is measurable. Suppose we measure an operator O having two eigenstates: one is Eq. (9.8) and the other is

$$\frac{1}{\sqrt{2}}|\text{undecayed}\rangle \otimes |\text{untriggered}\rangle \otimes |\text{unactivated}\rangle \otimes |\text{unbroken}\rangle \otimes |\text{live}\rangle - \frac{1}{\sqrt{2}}|\text{decayed}\rangle \otimes |\text{triggered}\rangle \otimes |\text{activated}\rangle \otimes |\text{broken}\rangle \otimes |\text{dead}\rangle \ .$$

We prepare the entangled state Eq. (9.8). If repeated measurements of O always yield Eq. (9.8), there is no collapse. So we can *in principle* verify the entangled state Eq. (9.8). However, we can then *in principle* play the following trick. After verifying the entangled state, we check whether the cat is alive or dead. If we find it dead, i.e. in the state

$$|\text{decayed}\rangle \otimes |\text{triggered}\rangle \otimes |\text{activated}\rangle \otimes |\text{broken}\rangle \otimes |\text{dead}\rangle \ ,$$

we again measure the operator O. A measurement of O leaves the cat in an eigenstate of O. The eigenstates of O are not orthogonal to the state in which the cat is alive, Eq. (9.7); so if we now check again whether the cat is alive or dead, we may find the cat alive. Hence if we can measure O, we can revive a dead cat.

That is, we can *in principle* verify an entangled state in the same sense that we can *in principle* reverse an irreversible process. Bohr took measurement to be an irreversible process [9]. For example, if a radium nucleus decays and marks a blank photographic film, that is an irreversible process. Suppose a blank film evolves by unitary evolution into a superposition of a blank film and a marked film. Can we verify the superposition? If so, then we can also turn marked film into blank film. Since we cannot in practice reverse this measurement, we cannot in practice verify the superposition. This statement holds not only for Schrödinger's cat and for photographic film, it holds for any measurement. If we cannot monitor the quantum phases of a superposition of states of a measuring device, the measuring device is in practice classical, as Bohr insisted, and the measurement is irreversible.

Today we can show, more explicitly than Bohr, why measurements are irreversible. Measurements are irreversible because they involve pointers and other macroscopic objects that

[4] Even direct measurements of the quantum wave (described in Chap. 15) would not prove or disprove any of these resolutions.

interact with the environment. Consider an entangled state of a measuring device and a measured system. The measuring device interacts with the environment and the relative phases of the entangled state come to depend on the environment. We can minimize this interaction, but we cannot eliminate it; consequently we lose track of the relative phases and cannot verify the entangled state. This form of irreversibility is called *decoherence*. In practice, decoherence happens so quickly that we cannot verify an entangled state of a measuring device and a measured system [10]. For all practical purposes, decoherence makes entanglement of a measuring device and a measured system unobservable; but decoherence cannot select the actual result of a measurement from the possible results.

If the paradox of Schrödinger's cat never arises in practice, why modify quantum mechanics? Perhaps some other theory accounts for the results of measurements. Perhaps boundary conditions account for them. The Many Worlds interpretation assumes initial boundary conditions on the universe (the initial state) but not final boundary conditions. In Sects. 10.5 and 18.3 we consider final boundary conditions as well. Can they account for the actual results of measurements?

Problems

9.1 (a) Show that the time evolution operator $U(T,0)$ defined in Eq. (9.5),

$$U(T,0) = e^{-i\pi\sigma_z\sigma_y^d/4} ,$$

acts on spin states as follows:

$$U(T,0)|\uparrow,0\rangle = |\uparrow,\downarrow\rangle$$
$$U(T,0)|\downarrow,0\rangle = |\downarrow,\uparrow\rangle .$$

(The initial state $|0\rangle_d$ of the "measuring device" is $(|\uparrow\rangle_d + |\downarrow\rangle_d)/\sqrt{2}$.) Thus

$$U(T,0)\frac{|\uparrow,0\rangle + |\downarrow,0\rangle}{\sqrt{2}} = \frac{|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle}{\sqrt{2}} .$$

(b) Consider an ensemble of spin pairs in the state $(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle)/\sqrt{2}$. Show that this ensemble is not equivalent to an ensemble that is a mixture of an equal number of pairs in the states $|\uparrow,\downarrow\rangle$ and $|\downarrow,\uparrow\rangle$. What would be the result of measuring the total spin of each pair in the two ensembles?

*9.2 In a Mach-Zehnder interferometer, an incident wave packet separates into two wave packets, which take different paths before interfering:

$$\psi(x,t) = \frac{1}{\sqrt{2}}[\psi_T(x,t) + \psi_R(x,t)] .$$

(See Fig. 9.2.) Since the evolution of $\psi(x,t)$ is unitary, we can write $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ and also $|\psi_T(t)\rangle = U(t)|\psi_T(0)\rangle$, $|\psi_R(t)\rangle = U(t)|\psi_R(0)\rangle$ where $U(t)$ is a unitary operator. Since $U(t)$ is unitary, the scalar product $\langle\psi_T(t)|\psi_R(t)\rangle$ must be constant. Yet since the two wave packets $\psi_T(x,t)$ and $\psi_R(x,t)$ separate, their overlap – and thus their scalar product $\langle\psi_T(t)|\psi_R(t)\rangle$ – must decrease at some time. Resolve this paradox!

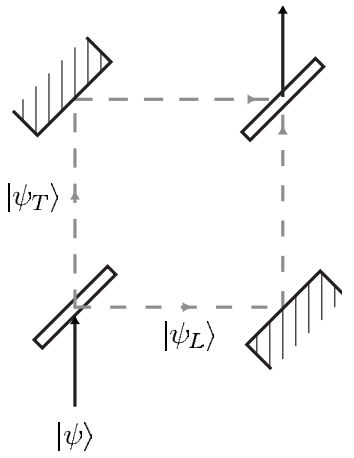

Figure 9.2: The wave packet $|\psi\rangle = (|\psi_T\rangle + |\psi_L\rangle)/\sqrt{2}$ is a superposition of $|\psi_T\rangle$, transmitted through the first mirror, and $|\psi_L\rangle$, reflected at the first mirror.

9.3 Suppose we interpret the wave function $\psi(\mathbf{x}, t)$ of a particle as follows: with probability density $\rho = |\psi(\mathbf{x}, t)|^2$ the particle is at $\mathbf{x}$ at time t and its velocity is

$$\mathbf{v}(t) = \frac{\hbar}{m} \nabla \left[\Im \ln \psi(\mathbf{x}, t) \right] \ ,$$

where $\Im$ indicates the imaginary part. Show that if $\psi(\mathbf{x}, t)$ satisfies the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right] \psi(\mathbf{x}, t) \ ,$$

then the probability density ρ and the probability current $\mathbf{J} = \rho \mathbf{v}$ satisfy the continuity equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \ .$$

*9.4 Apply the CSL model to the initial state Eq. (9.9). Define a state $|\Psi_w(t)\rangle$ with boundary condition $|\Psi_w(0)\rangle = |\Psi(0)\rangle$.
(a) Suppose the random field $w(\mathbf{x}, t)$ equals $2\lambda N$ (constant in time) over the undisplaced pointer position, and zero elsewhere. In the approximation $H = 0$, show that Eq. (9.10) implies

$$|\Psi_w(t)\rangle = \frac{1}{\sqrt{2}} \left[\bigotimes_i |\psi_0^{(i)}\rangle + e^{-t\lambda N^2 V/a^3} \bigotimes_i |\psi_L^{(i)}\rangle \right] \ , \tag{9.14}$$

where V is the volume of the pointer. The approximation $H = 0$ is natural when the integral term in Eq. (9.10) dominates the evolution of $|\Psi_w(t)\rangle$.
(b) Show that for iron (density 7.86 g/cm^3, predominant isotope ${}_{26}\mathrm{Fe}^{56}$) the number of particles (protons, neutrons and electrons) in a ball of radius $a = 10^{-5}$ cm is approximately $N = 2.9 \times 10^{10}$ particles. Thus, relative to the configuration $w(\mathbf{x}, t)$, the

probability of finding the pointer displaced by L becomes exponentially small in a time of order $T \approx a^3/\lambda N^2 V \approx 6 \times 10^{-18}$ sec.
(c) Now suppose that $w(\mathbf{x}, t)$ equals $w_0(t)$ all over the volume of the undisplaced pointer, and equals $w_L(t)$ all over the volume of the displaced pointer. Define times $t_n = nT/N$ for $n = 1, \ldots, N$ with N arbitrarily large. For each time interval $t_{n-1} \leq t < t_n$, define $w_0(t) = w_0(t_n)$ and $w_L(t) = w_L(t_n)$. The relevant measure of $w(\mathbf{x}, t)$ (up to overall normalization) is the product

$$\prod_{n=1}^{N} [dw_0(t_n) dw_L(t_n)] \ .$$

Compute the probability $\langle \Psi_w(T) | \Psi_w(T) \rangle$ and show that there are only two kinds of configurations $w(t)$ for which the probability is not exponentially suppressed: $w_0(t_n) \approx 2\lambda N$, $w_L(t_n) \approx 0$ and $w_0(t_n) \approx 0$, $w_L(t_n) \approx 2\lambda N$, for all n. Thus superpositions are suppressed.

*9.5 In the "Many Worlds" interpretation of quantum theory, consider measurements of a nondegenerate operator A, with eigenstates $|a_1\rangle, \ldots, |a_n\rangle$, on an ensemble of N systems prepared in the identical state $\sum_i c_i |a_i\rangle$. These measurements yield n^N incompatible accounts, each corresponding to a "world" with a certain measure. Within any world, let N_i denote the number of systems left in the state $|a_i\rangle$ by the measurement of A; by definition $\sum_i N_i = N$. Consider the combined measure of all the worlds in which $N_i/N = |c_i|^2$ to order $1/\sqrt{N}$, for all i, and show that in the limit $N \to \infty$, their combined measure approaches 1.

*9.6 Consider an ensemble of identical systems in which the probability that a system is in the state $|\psi_i\rangle$ is p_i. We represent this ensemble by a *density matrix* ρ:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \ .$$

(The states $|\psi_i\rangle$ need not be orthogonal.) Prove the following:
(a) On the ensemble, the expectation value of any operator O is the trace of ρO, $\mathrm{tr}(\rho O) = \sum_i p_i \langle\psi_i|O|\psi_i\rangle$.
(b) For every ρ, tr $\rho = 1$; but tr $(\rho^2) = 1$ only if $\rho = |\psi_i\rangle\langle\psi_i|$ for some i.
(c) Suppose we measure O on one system in the ensemble. By the definition of an ensemble, the density matrix of the other systems in the ensemble should remain ρ. Show that if tr $(\rho^2) = 1$ then the density matrix remains ρ, whether the number N of systems in the ensemble is finite or infinite. Show that the same is true if ρ represents Alice's systems when she and Bob share pairs of systems in an entangled state. (See Sect. 3.4.) Show that if ρ is a mixture of N_1 systems in the state $|\psi_1\rangle$, N_2 systems in the state $|\psi_2\rangle$, and so on, with $N_i/N = p_i$ for all i, then a measurement on one system changes the density matrix of the other systems unless N is infinite [11].

9.7 Consider an N-particle system in a "Schrödinger Cat" state $|\Psi_{SCAT}\rangle$:

$$|\Psi_{SCAT}\rangle = \frac{1}{\sqrt{2}} \left\{ |\psi_1^L, \psi_2^L, \ldots, \psi_N^L\rangle + |\psi_1^D, \psi_2^D, \ldots, \psi_N^D\rangle \right\} \ ,$$

where $|\psi_1^L, \ldots, \psi_N^L\rangle$ represents the "live cat" and its environment and $|\psi_1^D, \ldots, \psi_N^D\rangle$ represents the "dead cat" and its environment. The state $|\Psi_{SCAT}\rangle$ is normalized, but we do not assume $\langle\psi_i^L|\psi_i^D\rangle = 0$ for each i. A planned experiment to verify the relative phase in $|\Psi_{SCAT}\rangle$ (i.e. to distinguish $|\Psi_{SCAT}\rangle$ from the orthogonal state) goes awry when the N-th system escapes unmeasured from the laboratory. Show that the probability that measurements on the other $N-1$ states yield the correct relative phase is at most $\left[1 + |\langle\psi_N^L|\psi_N^D\rangle|\right]/2$.

*9.8 As an example of decoherence, consider a current of electrons passing through a quantum point contact [12] and interacting with a two-level system in the initial state $(|0\rangle+|1\rangle)/\sqrt{2}$. (A quantum point contact reflects a fraction of incident electrons just as a partially silvered mirror reflects a fraction of incident photons.) In the initial state of the electrons, $\sqrt{R}|R\rangle + \sqrt{T}|T\rangle$, the probability of reflection at the quantum point contact is R and the probability of transmission is T. The interaction of each electron with the two-level system is

$$\frac{(|0\rangle+|1\rangle)}{\sqrt{2}} \otimes (\sqrt{R}|R\rangle + \sqrt{T}|T\rangle) \rightarrow \sqrt{\frac{R}{2}}(|0,R\rangle+|1,R\rangle) + \sqrt{\frac{T}{2}}(|0,T\rangle + e^{i\phi}|1,T\rangle) \ ,$$

each transmitted electron inducing a relative phase ϕ between $|0\rangle$ and $|1\rangle$.
(a) Show that the uncertainty in the relative phase between the states $|0\rangle$ and $|1\rangle$ after N electrons arrive at the quantum point contact is roughly $2\sqrt{NT(1-T)}\phi$.
(b) If total dephasing occurs when the uncertainty in the relative phase is π, show that the dephasing time t, i.e. the time required for total dephasing, is $t = e\pi^2/4IT(1-T)\phi^2$, where I is the incident current.

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10 A Quantum Arrow of Time?

An arrow of time flies through our experience. We remember the past but not the future. We see balloons deflate – but never inflate – spontaneously; we scramble eggs and we cannot unscramble scrambled eggs. Many physical processes are not reversible. But almost all *elementary* physical processes *are* reversible. Certain exotic processes[1] involving the weak force are not [1], but there is no known connection between the irreversibility of these microscopic processes and the macroscopic irreversibility of our experience. So why is there an arrow of time?

One explanation of macroscopic irreversibility follows from a common formulation of the second law of thermodynamics: the entropy of a closed system never decreases over time – disorder never decreases. (See Sect. 1.3.) A macroscopic process that increases entropy is practically irreversible, and so irreversible macroscopic processes emerge from reversible microscopic processes. However, this explanation does not satisfy us completely. At the very least, it doesn't account for the order we find in the world. Without order, entropy could not increase, and there would be no irreversible processes.

We can explain the order by assuming an ordered initial state of the universe. But why assume an initial state? Why not assume a final state? By imposing an initial but not a final boundary condition, we have already sent the arrow of time flying. Suppose, instead, that we impose an intermediate condition: physical laws are time symmetric, and we specify the state of the universe at an intermediate time. If there is any uncertainty in the evolution of the state – due either to indeterminism in physical laws or to an incomplete specification of the state – there is no arrow of time. Whether we run time forwards or backwards, entropy ultimately increases. This argument (like the helium/neon paradox in Sect. 1.3) indicates that entropy does not define the arrow of time – it follows the arrow of time.

Irreversibility is at the heart of the paradox of Schrödinger's cat. (See Sect. 9.4.) Does entropy account for this irreversibility? What, indeed, is irreversible in a quantum measurement? Collapse is irreversible. Consider a quantum system S with orthonormal states $|s_i\rangle$ and a measuring device D in the initial state $|d_0\rangle$. As in Sect. 9.3, we let $|d_i\rangle$ represent the state of the measuring device when it indicates that S is in the state $|s_i\rangle$. The evolution

$$|s_i, d_0\rangle \to |s_i, d_i\rangle \ ,$$

for any i, defines the measurement. Now, if S is initially in a superposition $\sum_i c_i|s_i\rangle$, unitary evolution takes S and D to an entangled state:

$$\sum_i c_i|s_i, d_0\rangle \to \sum_i c_i|s_i, d_i\rangle \ . \tag{10.1}$$

[1] See Prob. 8.2.

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If the entangled state collapses, then the complete evolution is not Eq. (10.1) but

$$\sum_i c_i|s_i, d_0\rangle \to \sum_i c_i|s_i, d_i\rangle \to |s_j, d_j\rangle \ , \tag{10.2}$$

for some j with $c_j \neq 0$. The system and measuring device evolve to the state $|s_j, d_j\rangle$. Suppose now that S is initially in a *different* state $\sum_i c'_i|s_i\rangle$, with $c'_j \neq 0$. Assuming collapse, the system and measuring device could evolve to the same final state $|s_j, d_j\rangle$:

$$\sum_i c'_i|s_i, d_0\rangle \to \sum_i c'_i|s_i, d_i\rangle \to |s_j, d_j\rangle \ .$$

Since collapse can take two different initial states to the same final state, collapse is irreversible.

If collapse is irreversible, what is the relation of collapse to thermodynamic irreversibility? Does Eq. (10.2) entail an increase in entropy? If there is no increase in entropy, does quantum mechanics contain its own arrow of time? One way to approach these questions is to consider possible boundary conditions on quantum processes. We have so far imposed only an initial boundary condition on the measurement process. Why not impose a final boundary condition as well? This proposal may sound strange, but in practice, experiments often involve successive measurements of noncommuting observables. For example, if we measure the spin component of an atom successively along three distinct axes, the intermediate spin measurement is sandwiched between initial and final boundary conditions. Here classical and quantum mechanics differ fundamentally. The initial state of a classical system determines its final state (via its equations of motion). But in quantum mechanics, the initial state of a system does not determine its final state, and we can impose almost any combination of initial and final boundary conditions. So before concluding that quantum mechanics contains an arrow of time, let's check whether our boundary conditions contain an arrow of time – whether initial boundary conditions without final boundary conditions make reversible processes look irreversible.

10.1 A Quantum Card Trick

In a typical (classical) card trick, a magician offers a deck of cards to a volunteer. The volunteer picks a card at random, looks at it, and returns it at random to the deck, without showing it to the magician. Then the magician (call her Madge) identifies the card. How does she do it? She may, for example, prepare the cards in a certain order, and then identify the volunteer's card as the one that is out of order. Thus the protocol of the card trick is as follows: First, Madge prepares a state (the deck of cards). Second, the volunteer performs a measurement (picks a card, looks at it and returns it) without informing Madge of the result. Finally, Madge performs her own measurement (she gets back the deck and looks at it) and infers the result of the volunteer's measurement.

And now for a quantum card trick. A volunteer, please?

The protocol of the quantum card trick [2] is as follows: First, Madge prepares a state of a spin-1/2 particle. Next, you (the volunteer) perform a measurement on the particle. You may measure the spin component of the particle along the x-axis, the y-axis, or the z-axis – you choose which axis. You don't tell Madge which axis you chose, and you don't tell her the

result of your measurement. Finally, Madge performs a measurement of her own, and hands you a little list announcing the result of your measurement. Actually, the list announces three results, one for each allowed measurement. For example, Madge may hand you the following list:

If you measured σ_x, the result was 1.
If you measured σ_y, the result was -1.
If you measured σ_z, the result was -1.
$\star$ Have a nice day. $\star$

You may not believe it, but Madge is always right. Her list always includes the result of your measurement. How does she do it?

According to the principle of complementarity, measurements of σ_x, σ_y and σ_z on a particle at a given time are mutually exclusive. If Madge prepared the particle in an eigenstate of σ_x, she could predict your result if you chose to measure σ_x. Moreover, don't forget that Madge makes a final measurement, after yours. Suppose she measures σ_y. Then if you chose to measure σ_x, she could identify your result from the state she prepared; if you chose to measure σ_y, she could identify your result from her final measurement. But how could she identify your result if you chose to measure σ_z?

We leave this question as Prob. 10.1, with a hint: Madge could have an extra particle up her sleeve. No, she is not allowed to swap the particles; it is not an identical particle. But the initial state she prepares could be an entangled state of two particles.

10.2 Time Reversal

What is time reversal in quantum mechanics? Let's try to answer the question via an analogy with classical mechanics. Suppose $\mathbf{x}(t)$ represents the motion of a free classical particle, with $\mathbf{x}(t)$ the particle position at time t. The time reverse of the motion is $T\mathbf{x}(t) = \mathbf{x}(-t)$. The sequence of events is reversed in $T\mathbf{x}(t)$: if in $\mathbf{x}(t)$ the particle moves from $\mathbf{x}(t_a)$ to $\mathbf{x}(t_b)$ in time $t_b - t_a$, in $T\mathbf{x}(t)$ the particle moves from $\mathbf{x}(t_b)$ to $\mathbf{x}(t_a)$ in the same time.

By analogy, the time reverse of an evolving quantum state $\psi(\mathbf{x}, t)$ should be $\psi(\mathbf{x}, -t)$. But a simple example shows that this analogy cannot be correct: it would imply that the time reverse of a free particle in a momentum eigenstate,

$$\psi(\mathbf{x}, t) = e^{i\mathbf{p}\cdot\mathbf{x}/\hbar - ip^2 t/2m/\hbar} ,$$

is

$$\psi(\mathbf{x}, -t) = e^{i\mathbf{p}\cdot\mathbf{x}/\hbar + ip^2 t/2m/\hbar} ;$$

but both $\psi(\mathbf{x}, t)$ and $\psi(\mathbf{x}, -t)$ represent a particle with momentum $\mathbf{p}$, whereas time reversal should reverse the momentum. Furthermore, $\psi(\mathbf{x}, t)$ is a solution to the Schrödinger equation for a free particle,

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = \frac{p^2}{2m}\psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x}, t) ,$$

but $\psi(\mathbf{x}, -t)$ is not, and both $\psi(\mathbf{x}, t)$ and $T\psi(\mathbf{x}, t)$ should be solutions to the Schrödinger equation for a free particle, just as both $\mathbf{x}(t)$ and $T\mathbf{x}(t)$ could be motions of a free classical

particle. The analogy fails because the classical and quantum descriptions of a free particle are so different. All that $\mathbf{x}(t_0)$ tells us is the position of the particle at time t_0; the momentum depends on $d\mathbf{x}(t)/dt$ evaluated at $t = t_0$. By contrast, $\psi(\mathbf{x}, t_0)$ yields both the position and momentum distributions of a particle at a time t_0.

The position distribution depends on $\psi(\mathbf{x}, t_0)$ through the norm squared, and $|\psi(\mathbf{x}, t_0)|^2$ resembles $\mathbf{x}(t_0)$ in giving information only about the position distribution. For $|T\psi(\mathbf{x}, t)|^2$ to represent the time reverse of $|\psi(\mathbf{x}, t)|^2$, it must equal $|\psi(\mathbf{x}, -t)|^2$. Hence $T\psi(\mathbf{x}, t)$ and $\psi(\mathbf{x}, -t)$ can differ only in phase, for every $(\mathbf{x}, t)$. What is the phase? Let us write $\psi(\mathbf{x}, t)$ as a Fourier transform:

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3p \, \tilde{\psi}(\mathbf{p}, t) e^{i\mathbf{p}\cdot\mathbf{x}/\hbar} \, . \tag{10.3}$$

Taking the complex conjugate of Eq. (10.3), we see that the distribution of $\mathbf{p}$ in $\psi(\mathbf{x}, t)$ is identical to the distribution of $-\mathbf{p}$ in $\psi^*(\mathbf{x}, t)$; that is, taking the complex conjugate of $\psi(\mathbf{x}, t)$ reverses the direction of momentum, but leaves position unchanged. Thus, we may define the time reversal operator T by

$$T\psi(\mathbf{x}, t) = \psi^*(\mathbf{x}, -t) \, . \tag{10.4}$$

While this operator T acts appropriately on $\mathbf{x}$ and $\mathbf{p}$, we have not yet proved that T is the correct time reversal operator. (See Prob. 10.2.) However, we obtain the same time reversal operator T from the Schrödinger equation. Consider the Schrödinger equation for a single charged particle:

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = H(t)\psi(\mathbf{x}, t) \, , \tag{10.5}$$

where the Hamiltonian $H(t)$ depends explicitly on time:

$$H(t) = \frac{1}{2m} \left[-i\hbar\nabla - e\mathbf{A}(\mathbf{x}, t) \right]^2 + V(\mathbf{x}, t) \, .$$

Here $\mathbf{A}(\mathbf{x}, t)$ is the electromagnetic vector potential and $V(\mathbf{x}, t)$ includes the electric scalar potential. The state $\psi(\mathbf{x}, t)$ obeys Eq. (10.5) and evolves forward in time. Now assume that the state $T\psi(\mathbf{x}, t)$, too, evolves forward in time, but represents the reverse sequence of events. Then $T\psi(\mathbf{x}, t)$ must obey the Schrödinger equation, Eq. (10.5), with $H^T(t)$, the time reverse of $H(t)$, in the place of $H(t)$. What is $H^T(t)$? The time reverse of $H(t)$ must contain $-t$ wherever $H(t)$ contains t. In addition, it must contain $-\mathbf{A}$ instead of $\mathbf{A}$, because the vector potential $\mathbf{A}$ is a pseudovector. The sign of V, however, is unchanged under time reversal.[2] We obtain

$$H^T(t) = \frac{1}{2m} \left[-i\hbar\nabla + e\mathbf{A}(\mathbf{x}, -t) \right]^2 + V(\mathbf{x}, -t)$$

[2]The transformation of $\mathbf{A}$ and V follows from the transformation of $\mathbf{E}$ and $\mathbf{B}$, since $\mathbf{E} = \partial\mathbf{A}/\partial t - \nabla V$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Electric fields are produced by charges and thus remain the same under time reversal. Magnetic fields are produced by currents; since currents change sign under time reversal, so do magnetic fields. Thus $\mathbf{E}$ is a true vector while $\mathbf{B}$ is a pseudovector, i.e. $\mathbf{B}$ changes sign under time reversal.

and

$$i\hbar\frac{\partial}{\partial t}T\psi(\mathbf{x},t) = H^T(t)T\psi(\mathbf{x},t)\ . \tag{10.6}$$

If we now replace t by $-t$ everywhere in the original Schrödinger equation, Eq. (10.5), and also take the complex conjugate of both sides, we find

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi^*(\mathbf{x},-t) &= \frac{1}{2m}\left[i\hbar\nabla - e\mathbf{A}(\mathbf{x},-t)\right]^2\psi^*(\mathbf{x},-t) + V(\mathbf{x},-t)\psi^*(\mathbf{x},-t) \\ &= H^T(t)\psi^*(\mathbf{x},-t)\ . \end{aligned} \tag{10.7}$$

Comparing Eqs. (10.6–7), we find $T\psi(\mathbf{x},t) = \psi^*(\mathbf{x},-t)$, as above in Eq. (10.4).

Wigner, who first derived T, proved that T is an *anti*linear operator [3]. That is, if $\psi(\mathbf{x},t) = \alpha_1\psi_1(\mathbf{x},t) + \alpha_2\psi_2(\mathbf{x},t)$, then

$$\begin{aligned} T\left[\alpha_1\psi_1(\mathbf{x},t) + \alpha_2\psi_2(\mathbf{x},t)\right] &= T\psi(\mathbf{x},t) \\ &= \psi^*(\mathbf{x},-t) \\ &= \alpha_1^*\psi_1^*(\mathbf{x},-t) + \alpha_2^*\psi_2^*(\mathbf{x},-t) \\ &= \alpha_1^*T\psi_1(\mathbf{x},t) + \alpha_2^*T\psi_2(\mathbf{x},t)\ . \end{aligned}$$

But the proof that T is antilinear does not require assuming $T\psi(\mathbf{x},t) = \psi^*(\mathbf{x},-t)$. (See Prob. 10.4.) The definition of T in Eq. (10.4) is not always correct. Note that we can multiply T by an arbitrary phase factor to get an equivalent time reversal operator. In Eq. (10.4) we have set this phase factor to 1. However, we cannot always do so when T acts on states with spin. (See Sect. 11.2.)

10.3 The Aharonov–Bergmann–Lebowitz Formula

We know how to compute the probability of an outcome, given an initial boundary condition. Let the initial state at time t_a be $|a\rangle$; let $U(t,t_a)$ be the operator that evolves the state $|a\rangle$ forward in time from t_a to t. If at time t we measure a nondegenerate operator C having eigenvalues and eigenvectors c_j and $|c_j\rangle$, respectively, then $\mathcal{P}(c_j/a)$, the conditional probability of the result c_j, is $|\langle c_j|U(t,t_a)|a\rangle|^2$. However, our discussion of collapse and reversibility suggests that we need a formula for computing the probability of an outcome, given both initial *and* final boundary conditions. So does the quantum card trick.

Aharonov, Bergmann and Lebowitz derived such a formula [4]:

$$\mathcal{P}(c_j/a,b) = \frac{|\langle b|U(t_b,t)|c_j\rangle\langle c_j|U(t,t_a)|a\rangle|^2}{\sum_i|\langle b|U(t_b,t)|c_i\rangle\langle c_i|U(t,t_a)|a\rangle|^2}\ . \tag{10.8}$$

Here $|b\rangle$ is the final boundary condition at time t_b; $U(t_b,t)$ is the operator that evolves a state from t to time t_b; and $\mathcal{P}(c_j/a,b)$ is the probability that a measurement of C at time t yields c_j, given the boundary conditions $|a\rangle$ at time t_a and $|b\rangle$ at time t_b. The numerator of the Aharonov–Bergmann–Lebowitz (ABL) formula is the product of two conditional probabilities, the probability $\mathcal{P}(c_j/a)$ that a measurement of C at time t yields c_j, given that the state at time t_a is $|a\rangle$, and the probability $\mathcal{P}(b/c_j)$ that the state at time t_b is $|b\rangle$, given that a measurement

of C at time t yields c_j. To normalize the numerator, we divide it by the total probability $\sum_j \mathcal{P}(b,c_j)\mathcal{P}(c_j,a)$ that the state at time t_b is $|b\rangle$, given that the state at time t_a is $|a\rangle$, and assuming a measurement of C at time t. This total probability is the denominator of Eq. (10.8). The ABL formula allows us to compute the probabilities of results of any measurement between initial and final boundary conditions. For example, if $|a\rangle$ is an eigenstate of σ_x with eigenvalue 1, and $|b\rangle$ is an eigenstate of σ_y with eigenvalue -1, Eq. (10.8) shows that a measurement of σ_x at time t must yield 1, and a measurement of σ_y at time t must yield -1. A measurement of σ_z at time t is equally likely to yield 1 and -1.

Is the ABL formula invariant under time reversal? To answer this question, let us start with a Hamiltonian that is invariant under time reversal:

$$H = p^2/2m + V(x) \ .$$

The amplitude for a state $|a\rangle$ to evolve to a state $|b\rangle$ in a time $t_b - t_a$ is equal to

$$\langle b|U(t_b - t_a)|a\rangle \ ,$$

while the amplitude for the time-reversed process – for the state $|b\rangle^*$ to evolve to the state $|a\rangle^*$ in a time $t_b - t_a$, with the same Hamiltonian – equals

$$\begin{aligned}(\langle a|^*)U(t_b - t_a)(|b\rangle^*) &= (\langle a|U^*(t_b - t_a)|b\rangle)^* \\ &= (\langle a|U^*(t_b - t_a)|b\rangle)^\dagger \\ &= \langle b|U^*(t_a - t_b)|a\rangle \ .\end{aligned}$$

Now $U(t_b - t_a) = e^{-iH(t_b - t_a)/\hbar}$, so $U^*(t_a - t_b) = e^{iH^*(t_a - t_b)/\hbar} = U(t_b - t_a)$ since H is real. Hence the amplitudes are the same. In general, the ABL formula is invariant under time reversal if the relevant Hamiltonian is. (See Prob. 10.5.)

The ABL formula does not look time reversal invariant, only because the evolution operators $U(t_b,t)$ and $U(t,t_a)$ indicate the arrow of time. However, we can rewrite Eq. (10.8) to make it look time reversal invariant. From Eq. (5.14), we have

$$U^\dagger(t_j,t_i) = U(t_i,t_j) \ ,$$

Then Eq. (10.8) becomes

$$\mathcal{P}(c_j) = \frac{|\langle b|U^\dagger(t,t_b)|c_j\rangle\langle c_j|U(t,t_a)|a\rangle|^2}{\sum_i |\langle b|U^\dagger(t,t_b)|c_i\rangle\langle c_i|U(t,t_a)|a\rangle|^2} \ . \tag{10.9}$$

Since $\langle b|U^\dagger(t,t_b) = [U(t,t_b)|b\rangle]^\dagger$ we can regard $U^\dagger(t,t_b)$ in Eq. (10.10) as evolving $\langle b|$ *backwards* in time from t_b to t. Then the time evolution operators in Eq. (10.10) evolve the final state $\langle b|$ backward and the initial state $|a\rangle$ forward to the same intermediate time t, and no explicit arrow of time appears in Eq. (10.10). Actually, Eq. (10.8) can *always* be rewritten as Eq. (10.10), whether or not the relevant Hamiltonian is invariant under time reversal. What makes the ABL formula time reversal invariant (or not) is the Hamiltonian, and not the form of Eq. (10.8) or Eq. (10.10). However, Eq. (10.10) is convenient for problems with given initial and final boundary conditions, for it is simpler to evolve a final state $\langle b|$ backwards in time than to evolve several intermediate states $|c_i\rangle$ forwards in time.

Equation (10.8) and Eq. (10.10) assume one intermediate measurement, but there may be many such measurements. Let us consider a series of n intermediate measurements, at times $t_1, t_2, \ldots, t_n$, of nondegenerate operators $C^{(1)}, C^{(2)}, \ldots, C^{(n)}$, with eigenvalues $c_{i_1}^{(1)}, c_{i_2}^{(2)}, , \ldots, c_{i_n}^{(n)}$ respectively. (For degenerate operators, see Prob. 10.6.) We obtain the conditional probability $\mathcal{P}(c_{j_1}^{(1)}, c_{j_2}^{(2)}, \ldots, c_{j_n}^{(n)}/a, b)$ to find the values $c_{j_1}^{(1)}, c_{j_2}^{(2)}, \ldots, c_{j_n}^{(n)}$, respectively, for these operators:

$$\mathcal{P}(c_{j_1}^{(1)}, c_{j_2}^{(2)}, \ldots, c_{j_n}^{(n)}/a, b) \tag{10.10}$$
$$= \frac{|\langle b|U(t_b,t_n)|c_{j_n}^{(n)}\rangle \ldots \langle c_{j_2}^{(2)}|U(t_2,t_1)|c_{j_1}^{(1)}\rangle\langle c_{j_1}^{(1)}|U(t_1,t_a)|a\rangle|^2}{\sum_{i_1,i_2,\ldots,i_n} |\langle b|U(t_b,t_n)|c_{i_n}^{(n)}\rangle \ldots \langle c_{i_2}^{(2)}|U(t_2,t_1)|c_{i_1}^{(1)}\rangle\langle c_{i_1}^{(1)}|U(t_1,t_a)|a\rangle|^2} .$$

This expression is invariant under time reversal if the relevant Hamiltonian is. What about collapse in the ABL formula? Collapse appears explicitly in Eq. (10.11) at arbitrarily many times $t_1, t_2, \ldots, t_n$, since Eq. (10.11) is the conditional probability – not the probability amplitude – of the intermediate results $c_{j_1}^{(1)}, c_{j_2}^{(2)}, \ldots, c_{j_n}^{(n)}$. Hence, given both initial and final boundary conditions, collapse does not imply irreversibility; there is no "quantum arrow of time".

10.4 The Arrow of Time Revisited

The ABL formula, Eq. (10.8), assumes a final boundary condition, but how do we impose final boundary conditions? Final boundary conditions are no different from initial boundary conditions. We impose either initial or final boundary conditions by selecting for them. For example, to prepare an ensemble of particles with the initial boundary condition $S_x = \hbar/2$, we take a larger ensemble of the particles, measure S_x on each, and select particles with $S_x = \hbar/2$. If initial conditions alone define the ensemble, it is a *preselected* ensemble. To impose the final boundary condition $S_y = -\hbar/2$, we measure S_y on each particle – at the end of the experiment – and select only those with $S_y = -\hbar/2$. Together, initial and final conditions define a *pre- and postselected* ensemble. Pre- and postselected (PPS) ensembles often turn up in sequences of laboratory measurements.

The ABL formula applied to pre- and postselected ensembles (with Hamiltonians that are invariant under time reversal) provides a formulation of quantum mechanics that has no "arrow of time": all quantum processes are reversible. Is this formulation of quantum mechanics equivalent to the conventional formulation of quantum mechanics? We can (and did) derive this formulation from the conventional formulation; can we derive the conventional formulation from it? That is, can we derive the probability of a result, given initial boundary conditions alone (as in the conventional formulation)? At first it seems we cannot, because (by construction) the ABL formula yields the probability of a result given initial *and* final boundary conditions. However, we will now discover how to compute *both* kinds of probability.

To simplify, we drop all the time evolution operators in Eq. (10.11), i.e. we set them all to 1. (There is no loss of generality in doing so, since we can absorb them into the definitions of the states $|a\rangle, |c^{(1)j_1}\rangle, \ldots, |b\rangle$.) Now let C and C' be two noncommuting observables, and suppose a measurement of C yields the result c_j. What is the probability a measurement of C' will now yield the result c'_k? According to the conventional formulation of quantum mechanics, the

probability is $\mathcal{P}(c'_k/c_j) = |\langle c'_k|c_j\rangle|^2$. To obtain $\mathcal{P}(c'_k/c_j) = |\langle c'_k|c_j\rangle|^2$ from the ABL formula, we assume that C and C' are two successive intermediate measurements in Eq. (10.11), i.e. c'_k and c_j correspond to $c_k^{(m)}$ and $c_j^{(m-1)}$ for some m. We begin by applying Eq. (10.11) to calculate $\mathcal{P}(c'_k/a, c_j, b)$. Here $\mathcal{P}(c'_k/a, c_j, b)$ is the probability that a measurement of C' yields c'_k, given that the initial state is $|a\rangle$, the final state is $|b\rangle$, and that a measurement of C just prior to the measurement of C' yields c_j. (The intermediate measurements $C^{(1)}, \ldots, C^{(m-2)}$ and $C^{(m+2)}, \ldots, C^{(n)}$ are also given, although they are not explicit in the notation $\mathcal{P}(c'_k/a, c_j, b)$.) We apply Eq. (10.11) as follows: In the denominator, we sum over intermediate measurements of all operators *except* C alone. In the numerator, we sum over intermediate measurements of all operators except C *and* C'. We find that Eq. (10.11) no longer depends on any of the measurements that preceded C. We have

$$\begin{aligned}\mathcal{P}(c'_k/a, c_j, b) &= \frac{\sum_{j_{m+1}, j_{m+2}, \ldots, j_n} |\langle b|c_{j_n}^{(n)}\rangle \ldots \langle c_{j_{m+1}}^{(m+1)}|c'_k\rangle\langle c'_k|c_j\rangle|^2}{\sum_{i_m, i_{m+1}, \ldots, i_n} |\langle b|c_{i_n}^{(n)}\rangle \ldots \langle c_{i_m}^{(m)}|c_j\rangle|^2} \\ &= |\langle c'_k|c_j\rangle|^2 \frac{\sum_{j_{m+1}, j_{m+2}, \ldots, j_n} |\langle b|c_{j_n}^{(n)}\rangle \ldots \langle c_{j_{m+1}}^{(m+1)}|c'_k\rangle|^2}{\sum_{i_m, i_{m+1}, \ldots, i_n} |\langle b|c_{i_n}^{(n)}\rangle \ldots \langle c_{i_m}^{(m)}|c_j\rangle|^2} \ . \end{aligned} \tag{10.11}$$

$\mathcal{P}(c'_k/a, c_j, b)$ does not depend on any measurements prior to the measurement of C, but still depends on the measurements that follow C'. We might guess, however, that if the subsequent observables are sufficiently different from one another, they will tend to screen the influence of the final boundary condition; then the ratio of sums will *not* depend on the following measurements, and $\mathcal{P}(c'_k/a, c_j, b)$ will reduce to $|\langle c'_k|c_j\rangle|^2$, as in the conventional formulation.

Actually, our guess turns out to be correct, if we choose at least two of the observables correctly. Consider a sum that is part of both the numerator and the denominator of Eq. (10.12):

$$\sum_{j_{n-1}, j_n} |c_{j_{n-1}}^{(n-1)}\rangle\langle c_{j_{n-1}}^{(n-1)}|c_{j_n}^{(n)}\rangle\langle c_{j_n}^{(n)}|b\rangle\langle b|c_{j_n}^{(n)}\rangle\langle c_{j_n}^{(n)}|c_{j_{n-1}}^{(n-1)}\rangle\langle c_{j_{n-1}}^{(n-1)}| \ . \tag{10.12}$$

If the states $|c_{j_{n-1}}^{(n-1)}\rangle$ and $|c_{j_n}^{(n)}\rangle$ satisfy the condition that for all j_{n-1} and j_n,

$$|\langle c_{j_n}^{(n)}|c_{j_{n-1}}^{(n-1)}\rangle|^2 = \text{constant} \ ,$$

then the sum in Eq. (10.13) is a multiple of the identity matrix. (See Prob. 10.8.) Then the numerator and denominator of Eq. (10.12) both reduce to 1, and we obtain $\mathcal{P}(c'_k/a, c_j, b) = |\langle c'_k|c_j\rangle|^2$.

We can therefore obtain the conventional formula from the ABL formula by including suitable measurements in the sequence of measurements preceding the final boundary condition. If so, do we conclude that the ABL formula implies the conventional formulation? No, for if we can screen the effects of the final boundary conditions, we can also screen the effects of the initial boundary conditions. We obtained the result that if a measurement of C yields c_j, the subsequent measurement of C' will yield c'_k with probability $|\langle c'_k|c_j\rangle|^2$. This statement is a prediction. But the ABL formula has no arrow of time; we can just as well obtain from it the result that if a measurement of C' yields c'_k, the prior measurement of C *yielded* c_j with the *same* probability $|\langle c'_k|c_j\rangle|^2$. This statement is a retrodiction. In our experience, the prediction

is true but the retrodiction is false. Suppose, for example, we localize a photon in space. The subsequent momentum of the photon is uncertain, so we cannot say in what direction the photon will propagate; all directions are equally likely. But what about the direction of the photon's motion *before* the position measurement? Were all directions equally likely? No, the photon arrived from a lamp, a star, or from some other source of photons. This source of asymmetry in the direction of time is thermodynamic and cosmological; thermodynamics prescribes that physical systems that are not in thermal equilibrium tend towards equilibrium. If macroscopic physics did not provide this irreversibility, quantum mechanics could not itself generate it.

To conclude, the conventional formulation of quantum mechanics reflects the fact that predictions (as described above) are valid, while retrodictions are not. The ABL formula does not account for this fact and therefore does not imply the conventional formulation. By adding to the ABL formula the postulate that $\mathcal{P}(c'_k/a, c_j, b) = |\langle c'_k|c_j\rangle|^2$ is valid for predictions, but not retrodictions, we do obtain the conventional formulation, which therefore contains an arrow of time. The ABL formula, however, suggests that this arrow of time is not a *quantum* arrow of time; its origin is macroscopic rather than microscopic.

10.5 Boundary Conditions on the Universe

According to the ABL formula, Eq. (10.8), all quantum processes are reversible if the relevant Hamiltonian is reversible. Irreversibility is due to boundary conditions. What boundary questions? Are they symmetric under time reversal?

Cocke, Schulman, and Hartle and Gell-Mann [5] have considered boundary conditions on the universe that are symmetric in time. For example, let us consider a toy model universe consisting of a box filled with a nearly ideal gas. In the equilibrium state, the gas fills the box at constant density. But the initial state, at time $t = 0$, is far from equilibrium: all the molecules in the gas occupy one corner of the box, and are completely absent from the rest of the box. In the subsequent evolution, according to statistical mechanics, the gas approaches the equilibrium state. However, suppose we impose a final boundary condition at time $t = 2T$: we restrict the possible evolutions to those which lead, after a time $2T$, to a final state identical to the initial state. What is the evolution in this model?

With overwhelming probability the gas, with all the molecules initially in one corner, will diffuse more and more evenly throughout the box. After a time not less than T, however, we will see a strange phenomenon: the gas will begin to collect into the same corner of the box where it originally clustered, until finally at time $2T$ the gas is back in its initial state. We can distinguish three types of evolution, depending on T and the relaxation time. Let T_0 be the relaxation time of the initial state of the gas in the box; that is, without any special constraints on the final state, the gas would evolve from its initial configuration to equilibrium in a time T_0. For $T < T_0$, the gas will not reach equilibrium before it starts its reverse evolution towards the final state. For $T = T_0$, the gas will reach equilibrium but not remain there, whereas for $T > T_0$ the gas will reach equilibrium and remain there for a time $2(T - T_0)$ before evolving towards the final state. In each case, since the initial and final boundary conditions are identical, the evolution is invariant under time reversal. The three cases are depicted in Fig. 10.1, which shows results of a simulation of a finite number of particles and intermediate states. Figure 10.1 reveals an important difference between the two cases $T < T_0$ and $T \geq T_0$.

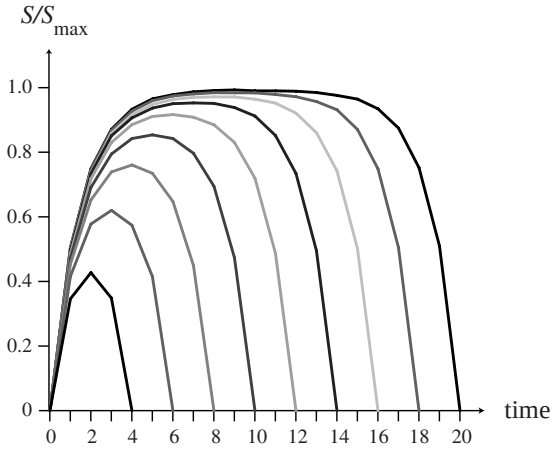

Figure 10.1: Simulations of a version of the Ehrenfest urn model, with four distinct balls distributed between two urns. At each time step either one ball or no ball (with equal probability) may jump from one urn to the other. The number of balls in each urn defines the macrostate, and the entropy of the macrostate is the logarithm of the number of arrangements of balls consistent with that macrostate. In the initial state, all the balls are in the same urn, and the final state is identical to the initial state. The graph shows entropy S divided by maximum entropy S_{max} as a function of time, averaged over many evolutions, for $2T$ ranging from 4 to 20 steps. The averages over many evolutions manifest time symmetry although individual evolutions do not. The relaxation time T_0 is approximately 8 steps. For $2T = 4$ to 12 steps ($T < T_0$), the evolution never reaches equilibrium (maximum entropy), while for $2T = 18$ to 20 steps ($T > T_0$), the evolution stays at equilibrium for a time and the final boundary condition is not evident in the evolution to equilibrium.

If T is less than T_0, the first half of the evolution is different from the evolution with no final boundary condition. If $T \geq T_0$, however, the first half of the evolution is indistinguishable from evolution with no final boundary condition. Only in the case $T \leq T_0$ could we detect, before the time T, evolution towards a final boundary condition.

Our discussion of final boundary conditions on the universe continues in Sect. 18.3, but without the assumption of time symmetry between initial and final boundary conditions. This assumption seems unwarranted. As far as we know, the universe initially had minimal entropy, and entropy has been increasing ever since. This increase in entropy is consistent with quantum mechanics. Under unitary time evolution, pure states remain pure states; but product states do not generally remain product states. Even if the state of the universe was initially a pure product state, and evolved according to unitary time evolution, it is today highly entangled; subsystems that once interacted locally have since separated to great distances. *Locally*, the state of each subsystem is not a pure state, but a mixture, and has entropy. Thus quantum mechanics offers an objective account of entropy increase, unrelated to "coarse graining": quantum entropy measures the entanglement of systems [6]. In particular, measurements leave measuring devices and measured systems entangled. On the one hand, systems that have interacted separate, increasing local entropy. On the other hand, unitary time evolution insures that no measurement result ever disappears without a trace. Measured values, carefully

recorded in laboratory notebooks, may still get lost; measuring devices may wear out; but under unitary evolution, the correlations among them remain.

Problems

*10.1 In Sect. 10.1, what operators could Madge measure in order to ascertain the result of your measurement?

*10.2 For a wave function $\psi(x)$ defined on a line, show that the probability distributions for position, $|\psi(x)|^2$, and for momentum, $|\tilde{\psi}(p)|^2$, do not determine $\psi(x)$. Here

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ \psi(x) e^{-ipx/\hbar}\ ,$$

*10.3 (a) Show that if T is antilinear and $|\psi(t)\rangle$ and $T^2|\psi(t)\rangle$ represent the same physical state, then $T^2|\psi(t)\rangle = \pm|\psi(t)\rangle$.
(b) For T and $H^T(t)$ in Eqs. (10.4) and (10.6), show that $H^T(t) = TH(t)T$.

10.4 Consider a time reversal operator T and a system with Hamiltonian H such that $[H, T] = 0$. Let $\psi(\mathbf{x}, 0)$ be an arbitrary state of the system.
(a) Explain, without assuming $T\psi(\mathbf{x}, t) = \psi^*(\mathbf{x}, -t)$, why $e^{iHt/\hbar}T\psi(\mathbf{x}, 0)$ and $Te^{-iHt/\hbar}\psi(\mathbf{x}, 0)$ represent the same physical state.
(b) Show that T cannot be a linear operator.
(c) Let P represent parity. Would a version of this argument show that P cannot be a linear operator?

10.5 Show that if the Hamiltonian H of a system commutes with T, then the Hilbert space of the system has a basis of real eigenvectors of H.

10.6 Equations (10.8) and (10.10) apply to the measurement of an operator C with nondegenerate eigenstates $|c_i\rangle$. But if C has degeneracies, a generalized ABL formula [7] applies. Show how to generalize the formula by replacing the projectors $|c_i\rangle\langle c_i|$, etc., in the numerator and denominator of Eqs. (10.8) and (10.10) by projectors onto the degenerate subspaces of C.

10.7 Consider the ABL formula, Eq. (10.11), modified as follows: in the numerator, we sum over all intermediate measurements $C^{(1)}, C^{(2)}, \ldots, C^{(n)}$ except for consecutive measurements of observables C and C', where $C = C^{(m-1)}$ and $C' = C^{(m)}$ for some m. In the denominator, we sum over the results of all intermediate measurements except of the measurement of C alone. Show that the ratio of the numerator and the denominator does not depend on the measurements that preceded C. (See also Eq. (10.12).)

10.8 (a) Consider two observables, R and S, on the same Hilbert space of dimension n; the eigenstates of R are $|r_1\rangle, |r_2\rangle, |r_3\rangle, \ldots, |r_n\rangle$ and the eigenstates of S are $|s_j\rangle$, where

$$|s_j\rangle = n^{-1/2} \sum_{k=1}^{n} e^{ijk\pi/n} |r_k\rangle\ .$$

Show that the eigenstates $|s_j\rangle$ constitute an orthonormal basis if the $|r_j\rangle$ do and that for every j and j',

$$|\langle r_j|s_{j'}\rangle|^2 = 1/n \ .$$

(b) Show that for any normalized state $|b\rangle$, the sum

$$\sum_{j,j\prime} |r_j\rangle\langle r_j|s_{j'}\rangle\langle s_{j'}|b\rangle\langle b|s_{j'}\rangle\langle s_{j'}|r_j\rangle\langle r_j|$$

equals I/n where I is the identity matrix.

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11 Superselection Rules

Our claim, throughout Chaps. 7–10, is that nonrelativistic quantum mechanics corresponds completely with experiment: not only does the theory correctly predict the results of experiments, it predicts *just* the results of possible experiments, neither more nor less. But many physicists claim the contrary. Wick, Wightman and Wigner claimed, in 1952, that some Hermitian operators of nonrelativistic quantum mechanics can never be measured; thus theory and experiment do not correspond completely [1]. They introduced the term *superselection rule*; each superselection rule asserts that some Hermitian operator of quantum theory is not an observable. Such a rule would invalidate our treatment of quantum measurements, because we assume (following von Neumann) that any Hermitian operator is measurable. Since the treatment of quantum measurements is central to this book, we must show that there are no superselection rules in nonrelativistic quantum theory. (But in *relativistic* quantum theory, there certainly are superselection rules; see Prob. 11.1 and Chap. 14.)

The argument of Wick, Wightman and Wigner uses the time reversal operator T of Sect. 10.2. Additional properties of T appear in Sect. 11.2, and Sect. 11.3 presents the Wick–Wightman–Wigner argument. But first, Sect 11.1 presents a paradox concerning a measurement of angular momentum. According to this paradox, angular momentum is not measurable. Yet there is no superselection rule for angular momentum! After resolving this paradox, in Sect. 11.4, we can resolve the argument about superselection rules.

11.1 Superselection Rule for Angular Momentum?

Imagine a closed, free-floating and isolated laboratory, with a Stern-Gerlach apparatus rigidly mounted in it. The laboratory contains an isolated atom and we would like to measure the spin component of the atom along the z-axis of Fig. 11.1. We measure S_z with the Stern-Gerlach apparatus: the atom enters the apparatus perpendicular to the field gradient, and if it deflects up, we say S_z is positive; if the atom deflects down, we say S_z is negative. Although this is the standard way to measure the spin component of an atom, someone might criticize our measurement as follows:

> You cannot possibly measure the spin component of the atom. You and your laboratory and everything in it, including the atom, do not interact with anything else in the universe during the experiment. Thus, the total angular momentum of the laboratory and its contents is a conserved quantity. In particular, the total angular momentum along the x-axis is a conserved quantity throughout the experiment. Now, the total

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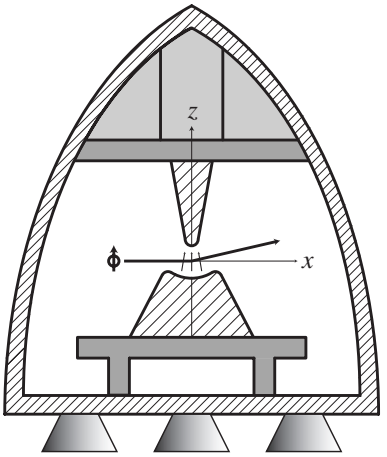

Figure 11.1: A Stern-Gerlach measurement (as in Fig. 7.2) inside a space capsule. Note that the thrusters (rocket engines) at the base of the capsule are not in use.

angular momentum along the x-axis is

$$J_x = S_x + L_x + J_x^{lab} \ ,$$

where S_x is the x-component of the spin of the atom, L_x is the x-component of the orbital angular momentum of the atom and J_x^{lab} is the angular momentum of everything else in the laboratory. Clearly S_z, the z-component of the spin of the atom, does not commute with J_x, because it does not commute with S_x. So you could not have measured it!

We might not accept this argument but there is something reasonable in it. According to von Neumann, we can measure S_z with an interaction Hamiltonian of the form

$$H_{int} = g(t) S_z P_d \ ,$$

where P_d is conjugate to a "pointer" displacement Q_d that shows the result of the measurement. (See Sect. 7.2.) The total Hamiltonian H is then

$$H = H_{int} + H_{atom} + H_{lab} \ ,$$

where $H_{int} = g(t) S_z P_d$ applies to the measurement, H_{atom} applies to the atom and H_{lab} applies to everything else in the laboratory. Now H and J_x do not commute, because $[H, J_x]$ contains the term $[H_{int}, S_x] = i\hbar g(t) S_y P_d$ and nothing else in $[H, J_x]$ can cancel this term. If H contains H_{int}, then S_x is uncertain and so J_x is conserved only approximately. But J_x is exactly conserved!

This paradox is not limited to angular momentum. If von Neumann's approach is wrong for this measurement, it is wrong for many other measurements, as well. Any measurement interaction that does not commute with *each* additive conserved quantity of the total system (the measured system and measuring device) will lead to a paradox. For example, suppose we want to measure the position of a particle in one space dimension. Let x denote the position of the particle and p its momentum, and let P denote the momentum of the measuring device.

If the particle and measuring device are free and isolated, apart from their mutual interaction, then the total momentum $p+P$ is a conserved quantity. But x does not commute with the total momentum:

$$[p+P,x] = [p,x] = -i\hbar \ .$$

Apparently, either we cannot measure the position of the particle, or else $p+P$ cannot be a conserved quantity. A measurement of x makes p uncertain, and then there is no sense in saying that $p+P$ does not change during the measurement.

11.2 T and Spin

The argument of Wick, Wightman and Wigner involves the time reversal operator T acting on quantum spin states. Section 10.2 introduces T as an antilinear operator, and Eq. (10.4) defines its action:

$$T\psi(\mathbf{x},t) = \psi^*(\mathbf{x},-t) \ .$$

However, Eq. (10.4) does not define the action of T on states with spin. What does time reversal do to spin?

Spin is a form of angular momentum, so we can infer the action of T on spin states from its action on orbital angular momentum states [2]. Time reversal makes a spinning object spin in the opposite sense, with the same angular speed. So the time reverse of a state of angular momentum $m\hbar$ along the z-axis, for example, should be a state with angular momentum $-m\hbar$ along the z-axis. Is it? Eigenstates of L_z, the z-component of orbital angular momentum, are proportional to $e^{im\phi}$, where ϕ is the azimuthal angle about the z-axis and $m\hbar$ is the angular momentum. According to Eq. (10.4), T sends $e^{im\phi}$ to $e^{-im\phi}$, so indeed T replaces $m\hbar$ with $-m\hbar$. Writing the eigenstates of L_z as kets $|m\rangle$, we have

$$T|m\rangle = |-m\rangle \ . \tag{11.1}$$

Here, as in Eq. (10.4), T could include an arbitrary overall phase factor. We have set it to 1 for convenience.

We soon discover, however, that we *cannot* always set the phase to 1. Consider a spin-1/2 particle. The eigenvalues of $S_z = \hbar\sigma_z/2$ are $\hbar/2$ and $-\hbar/2$. The corresponding eigenstates are $|1/2\rangle$ and $|-1/2\rangle$ in the notation of Eq. (11.1), or $|\uparrow\rangle$ and $|\downarrow\rangle$ in our usual notation. Applied to $|\uparrow\rangle$ and $|\downarrow\rangle$, Eq. (11.1) yields

$$T|\uparrow\rangle = |\downarrow\rangle \ , \qquad T|\downarrow\rangle = |\uparrow\rangle \ . \tag{11.2}$$

But Eq. (11.2) implies that

$$T\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \ . \tag{11.3}$$

Now $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ is an eigenstate of S_x with eigenvalue $\hbar/2$; according to Eq. (11.3), T does nothing to S_x. But T must reverse the direction of S_x just as it reverses the direction

of S_z. Thus Eq. (11.1) cannot define the action of T correctly. Note that it does not help to include in T a phase factor that is the same for all spin states; T cannot reverse the direction of S_x unless it multiplies $|\uparrow\rangle$ and $|\downarrow\rangle$ by different phases.

Once we realize that T must include a phase that depends on the spin state, it is not hard to discover the action of T. We write

$$T|\uparrow\rangle = e^{i\alpha_\uparrow}|\downarrow\rangle\ , \qquad T|\downarrow\rangle = e^{i\alpha_\downarrow}|\uparrow\rangle\ . \tag{11.4}$$

Then T reverses the direction of S_x if and only if

$$e^{i(\alpha_\uparrow - \alpha_\downarrow)} = -1\ . \tag{11.5}$$

If T satisfies Eq. (11.5), and *if* T is an antilinear operator, then it also reverses the direction of S_y. Indeed, T reverses the spin of any spin-1/2 state. (See Prob. 11.2.)

Now comes a little surprise. Suppose we act on the state $|\uparrow\rangle$ *twice* with T. We find

$$\begin{aligned} T^2|\uparrow\rangle &= Te^{i\alpha_\uparrow}|\downarrow\rangle \\ &= e^{-i\alpha_\uparrow}T|\downarrow\rangle \\ &= e^{-i\alpha_\uparrow}e^{i\alpha_\downarrow}|\uparrow\rangle \\ &= -|\uparrow\rangle\ , \end{aligned} \tag{11.6}$$

applying Eq. (11.5). Acting on the state, T^2 reverses the direction of time and then reverses it again – so it should leave the state unchanged. We might have expected $T^2 = 1$, but actually T^2 could equal any phase factor, since an overall phase does not change a physical state. (See also Prob. 10.3.) What matters is that T^2 should equal the *same* phase factor for all physical states – otherwise T^2 would change some superpositions of states. Computing $T^2|\downarrow\rangle$ as in Eq. (11.6), we find that $T^2|\downarrow\rangle = -|\downarrow\rangle$; hence $T^2 = -1$ for all spin-1/2 states, and two time reversals leave any state unchanged.

And now comes a big surprise. For states $|m\rangle$ with $S_z = m\hbar$, where m is an integer, we assumed Eq. (11.1). But since T may include an m-dependent phase, we must modify Eq. (11.1):

$$T|m\rangle = e^{i\alpha_m}|-m\rangle\ .$$

In particular, $T|0\rangle = e^{i\alpha_0}|0\rangle$. If we ask how T^2 acts on $|0\rangle$, we find $T^2|0\rangle = T(e^{i\alpha_0}|0\rangle) = e^{-i\alpha_0}T|0\rangle = |0\rangle$. So $T^2 = 1$ here! Of course if $T^2|0\rangle = |0\rangle$, we must have $T^2|m\rangle = |m\rangle$ for all $m = 0, \pm1, \pm2, \ldots$, otherwise T^2 would change some linear combinations of these states. We can insure $T^2 = 1$ for all the $|m\rangle$ with integer m by assuming $e^{i\alpha_{-m}} = e^{i\alpha_m}$ for all integer m. However, we cannot define T^2 so that it acts uniformly on all spin states, both when m is an integer and when it is not; T^2 does not act uniformly even on $|0\rangle$ and $|\uparrow\rangle$.

11.3 The Wick–Wightman–Wigner Argument

The argument of Wick, Wightman and Wigner is now clear. Two applications of time reversal must leave all physical states unchanged. But T^2 applied to a state such as $\big(|0\rangle + |\uparrow\rangle\big)/\sqrt{2}$ does *not* leave the state unchanged:

$$T^2 \frac{1}{\sqrt{2}}\big(|0\rangle + |\uparrow\rangle\big) = \frac{1}{\sqrt{2}}\big(|0\rangle - |\uparrow\rangle\big) \; .$$

According to quantum mechanics, the states $|\Psi_+\rangle \equiv \big(|0\rangle + |\uparrow\rangle\big)/\sqrt{2}$ and $|\Psi_-\rangle \equiv \big(|0\rangle - |\uparrow\rangle\big)/\sqrt{2}$ are physically distinct. But according to the Wick–Wightman–Wigner argument, they are not. If $|\Psi_+\rangle$ and $|\Psi_-\rangle$ are not physically distinct, then any operator connecting $|0\rangle$ and $|\uparrow\rangle$ must be unmeasurable. For example, define an operator A:

$$A = |0\rangle\langle\uparrow| + |\uparrow\rangle\langle 0| \; . \tag{11.7}$$

The expectation values of A in the states $|\Psi_+\rangle$ and $|\Psi_-\rangle$ are not the same. Indeed, $|\Psi_+\rangle$ and $|\Psi_-\rangle$ are eigenstates of A with eigenvalues 1 and -1, respectively. If A is measurable, then $|\Psi_+\rangle$ and $|\Psi_-\rangle$ are physically distinct. Thus A must be unmeasurable, and a relative phase between $|0\rangle$ and $|\uparrow\rangle$ has no physical meaning.

We have seen that $T^2 = 1$ for any state $|m\rangle$ with integer m, while $T^2 = -1$ for any state $|m\rangle$ with $m = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$. We refer to states with $T^2 = 1$ as *boson* states, and states with $T^2 = -1$ as *fermion* states, in accordance with the spin-statistics theorem [3]. (This theorem shows an intimate connection among rotation, time reversal and interchange of particles in quantum theory.) Thus $T^2 = 1$ for bosons and $T^2 = -1$ for fermions. According to Wick, Wightman and Wigner, any operator connecting boson and fermion states is unmeasurable.

There is another argument in favor of this superselection rule. A rotation of 2π leaves boson states unchanged, but gives fermion states a phase of -1. We can obtain the phase from the operator for rotations. Let a unit vector $\mathbf{n}$ define the direction of an axis passing through a particle, and let $\mathbf{S}$ be the spin operator for the particle. The operator $e^{-i\mathbf{S}\cdot\mathbf{n}\theta/\hbar}$ rotates the state of the particle an angle θ about the axis. Since the eigenvalues of $\mathbf{S}\cdot\mathbf{n}$ are $m\hbar$, where $m = \pm 1/2, \pm 3/2, \ldots$ for fermions and m is an integer for bosons, a rotation of 2π gives any fermion state a phase of -1 but leaves boson states unchanged. Suppose we prepare the state $|\Psi_+\rangle$. It is an eigenstate of the operator A defined in Eq. (11.7), and $A|\Psi_+\rangle = |\Psi_+\rangle$. A 2π rotation transforms $|\Psi_+\rangle$ into $|\Psi_-\rangle$, and $A|\Psi_-\rangle = -|\Psi_-\rangle$. Wick, Wightman and Wigner claimed that A is unmeasurable since a rotation through 2π should not change the value of any observable.

11.4 Everything is Relative

The paradoxes of Sect. 11.1 force us to think about what we actually measure. One paradox states that if the total momentum of a particle and a measuring device, $p + P$, is a conserved quantity, then we cannot measure x, the position of the particle, because it does not commute with the total momentum. But what is x? The absolute position of the particle? We cannot

measure the absolute position of the particle any more than we can measure the absolute position of the universe. What we actually measure is the position of the particle relative to the position X of the measuring device, i.e. we measure $x - X$; the relative position *does* commute with the total momentum $p + P$,

$$[p + P, x - X] = 0 \ ,$$

so we can measure position despite conservation of momentum.

The solution of the paradox of angular momentum is just as simple. We cannot measure absolute quantities. Do we really measure S_z? What defines the z-axis? We measure the component of angular momentum of the particle along an axis attached to the laboratory, not along an axis external to the laboratory. The Stern-Gerlach device measures $\mathbf{S} \cdot \mathbf{e}_z$, where $\mathbf{e}_z$ is a unit vector along the axis of magnet in the device. (See Fig. 11.2.) The question is whether $\mathbf{S} \cdot \mathbf{e}_z$, the angular momentum component along the axis of the magnet, commutes with J_x. To see that it does, we define $\mathbf{e}_z$ with respect to an external frame, in spherical coordinates:

$$\mathbf{e}_z = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \ .$$

In the same spherical coordinates, L_x is

$$L_x = i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi \frac{\partial}{\partial\phi} \right) \ ,$$

and we require

$$0 = [J_x \ , \ \mathbf{S} \cdot \mathbf{e}_z] = \left[L_x + \frac{\hbar}{2}\sigma_x \ , \ \frac{\hbar}{2}\sigma \cdot \mathbf{e}_z \right] \ . \tag{11.8}$$

It is straightforward to check that Eq. (11.8) holds. The lesson is that all measurements refer to relative quantities. Even the time in a coupling $g(t)$ is relative time; to be explicit, we can write $t - t_d$ instead of t, where t_d is a reference time.

If we now return to the Wick, Wightman and Wigner argument for a boson-fermion superselection rule, we find it resembles these paradoxes. A rotation of 2π introduces a relative phase of -1 between boson and fermion states; thus it takes a state such as $(|0\rangle + |\uparrow\rangle)/\sqrt{2}$ to an orthogonal state. Yet a 2π rotation should leave physical states unchanged. Now we stop and ask: a 2π rotation with respect to what? On the one hand, a 2π rotation of the whole universe leaves physical states unchanged. Indeed, a rotation of the universe through *any* angle leaves physical states unchanged, because there is no meaning to rotation unless it is rotation with respect to something that does not rotate. On the other hand, a *relative* rotation through 2π could in principle change physical states. True, in classical physics a relative rotation through 2π does not change any physical state, but quantum physics is not classical physics.[1] Thus, we claim that the argument for a boson-fermion superselection rule confuses absolute and relative. The rule only forbids measuring operators that we anyway cannot measure, because they represent absolute quantities.

[1] Even in classical physics, a 2π rotation of one object with respect to another can change their physical state, if the two objects are linked. For example, fix one end of a long ribbon and hold the free end near the fixed end. A complete rotation of the free end twists the ribbon. But *another* complete rotation removes the twist: just loop the middle of the ribbon around the free end.

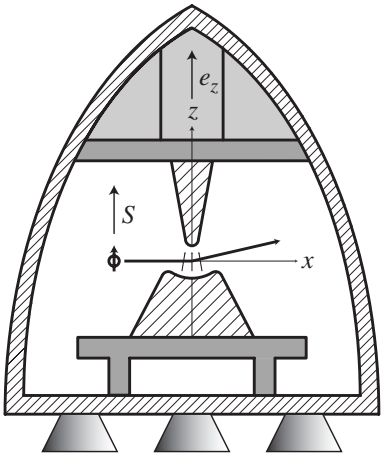

Figure 11.2: The Stern-Gerlach measurement as in Fig. 11.1 showing the axes of the spin and of the magnet.

We can see the effect of a relative 2π rotation in a thought experiment on a neutron [4]. Consider two identical boxes, with a constant magnetic field inside each. (See Fig. 11.3.) The magnetic fields are equally strong, and initially parallel. We will call one box the "nonrotating" (NR) box and the other the "rotating" (R) box; the latter is free to rotate around an axis perpendicular to the direction of the magnetic field. We can also separate the boxes or bring them into contact; when they are in contact, we can open a small window in their common side. Suppose that when the boxes are apart, the neutron Hamiltonian is H_0 and the neutron ground states $|NR\rangle$ and $|R\rangle$ have the same energy E, i.e. H_0 is degenerate; but when the two boxes are in contact with the window open, a small interaction term H_{int} allows the neutron to pass from one box to the other and breaks the degeneracy. Then in addition to $H_0|NR\rangle = E|NR\rangle$ and $H_0|R\rangle = E|R\rangle$ when the boxes are apart, we have $H_{int}|R\rangle = -\epsilon|NR\rangle$ and $H_{int}|NR\rangle = -\epsilon|R\rangle$ when the boxes are in contact and the window open. The eigenstates of $H_0 + H_{int}$ are not $|NR\rangle$ and $|R\rangle$ but rather the superpositions

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|NR\rangle + |R\rangle\right) \ , \quad |-\rangle = \frac{1}{\sqrt{2}}\left(|NR\rangle - |R\rangle\right) \ , \tag{11.9}$$

with energies $E_+ = E - \epsilon$ and $E_- = E + \epsilon$ respectively. The interaction makes the neutron tunnel between the boxes. Suppose that at time $t = 0$ we prepare the neutron in the state $|NR\rangle$, which equals $(|+\rangle + |-\rangle)/\sqrt{2}$. With the boxes in contact and the window open, the state of

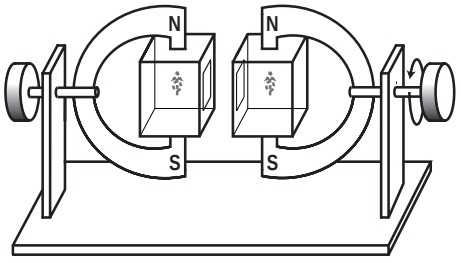

Figure 11.3: An apparatus to rotate, adiabatically, one neutron wave packet relative to another, for the thought experiment of Sect. 11.4.

the neutron at time $t > 0$ is

$$\frac{1}{\sqrt{2}}\left[e^{-iE_+t/\hbar}|+\rangle + e^{-iE_-t/\hbar}|-\rangle\right] ,$$

and the probability to find the neutron in the state $|NR\rangle$ at time t is $[1 + \cos(2\epsilon t/\hbar)]/2 = \cos^2(\epsilon t/\hbar)$. That is, the neutron is certainly in the state $|NR\rangle$ at times $t = nh/2\epsilon$ and certainly in the state $|R\rangle$ at times $t = nh/2\epsilon + h/4\epsilon$. The tunnelling current is

$$\frac{d}{dt}\cos^2(\epsilon t/\hbar) = -\frac{\epsilon}{\hbar}\sin(2\epsilon t/\hbar) ,$$

and its magnitude is maximal at times $t = nh/2\epsilon \pm h/8\epsilon$, changing direction every $h/4\epsilon$.

So far we have not rotated the "rotating" box. To demonstrate the effect of a relative 2π rotation, we wait until the tunnelling into the state $|R\rangle$ is maximal, e.g. until time $t = h/8\epsilon$, then close the window and separate the boxes. Assume that we cut off the interaction instantly, so the tunnelling stops just when the neutron is equally likely to be in either box. We now effect a slow 2π rotation of the rotating box. During the rotation, the Hamiltonian for the state $|R\rangle$ is

$$H_R(t) = e^{-i(\mathbf{L}+\mathbf{S})\cdot\mathbf{n}\omega t/\hbar} H_0 e^{i(\mathbf{L}+\mathbf{S})\cdot\mathbf{n}\omega t/\hbar} , \tag{11.10}$$

where $\mathbf{L}$ is the angular momentum of the neutron, $\mathbf{S}$ is its spin, $\mathbf{n}$ is a unit vector along the axis of rotation, and ω is the angular frequency of rotation. The instantaneous ground state of H_R is

$$e^{-i(\mathbf{L}+\mathbf{S})\cdot\mathbf{n}\omega t/\hbar}|R\rangle , \tag{11.11}$$

and a 2π rotation (which lasts a time $t = 2\pi/\omega$) turns the state $|R\rangle$ into $-|R\rangle$, since the eigenvalues of $\mathbf{L}\cdot\mathbf{n}/\hbar$ are integers and the eigenvalues of $\mathbf{S}\cdot\mathbf{n}/\hbar$ are $\pm 1/2$. Only the state $|R\rangle$ changes sign; thus the relative phase between $|NR\rangle$ and $|R\rangle$ changes by π. In addition, there is a contribution to the relative phase due to the different energies of the neutron in the two boxes (when one box is rotating). However, this contribution vanishes in the limit $\omega \to 0$. Since we can make the rotation arbitrarily slow, we neglect this contribution.

Wick, Wigner and Wightman did not consider rotating $|R\rangle$ relative to $|NR\rangle$. The rotation sends $|+\rangle$ to $|-\rangle$ and vice versa. If we now bring the boxes into contact and open the window, we will find that the tunnelling into the state $|NR\rangle$ (and not $|R\rangle$) is maximal, because we have reversed the relative phase between $|+\rangle$ and $|-\rangle$. This change in the tunnelling current would (in principle) be observable.

Hence a relative 2π rotation yields an observable phase. We can express either $|NR\rangle$ or $|R\rangle$ as a superposition of a fermion and a boson – a fermion if the neutron is in the corresponding box, a boson if it is not. (See Prob. 11.6.) A 2π rotation changes the relative phase of the fermion and boson; but to observe this phase we must rotate one box relative to the other box. A relative double time reversal would produce the same observable phase, except that we cannot effect a relative double time reversal as we can effect a relative 2π rotation.

11.5 Superposing Charge States

Wick, Wightman and Wigner also proposed a superselection rule for charge. A charge superselection rule asserts that, in a superposition of states $|n\rangle$ of different charges,

$$\sum_n c_n |n\rangle \ , \tag{11.12}$$

only the absolute values of the coefficients c_n, and not their phases, are measurable. The argument for this rule draws on quantum field theory. The terms appearing in a quantum field theory Lagrangian are products of fields Φ_j, one field for each particle. Each term must conserve charge; an interaction term may create and destroy charged particles but must not alter the net charge. As a consequence, each term has the following property: if we multiply every field Φ_j by $e^{i\theta Q_j}$, where Q_j is the charge of the field (i.e. of the particles associated with the field) and θ is a constant, then these phases cancel in each term; the terms do not acquire phases. Thus every quantum field theory is invariant under $e^{i\theta Q}$, where

$$e^{i\theta Q}\Phi_j = e^{i\theta Q_j}\Phi_j \ .$$

It is plausible that the operator $e^{i\theta Q}$ leaves physical states invariant, too. However, it does not leave Eq. (11.12) invariant. If $e^{i\theta Q}$ leaves physical states invariant, then the relative phases in Eq. (11.12) cannot be measurable.

Here, too, we must ask what we expect to measure. Applied to the state of the whole universe, $e^{i\theta Q}$ induces an absolute change in phase that we cannot expect to measure. However, we can expect to measure changes of phase in one state relative to a reference state that does not change. The challenge, then, is to build a suitable reference state. Let us build a reference state $|z\rangle$ from a coherent superposition of mesons [5]:

$$|z\rangle = \sum_n \frac{z^n}{(n!)^{1/2}} |n\rangle \ . \tag{11.13}$$

Here $|n\rangle$ is a state with n mesons, and z is a complex number. The mesons are positively charged; a neutron absorbs one to become a proton, or a proton emits one to become a neutron. We prepare the state $|z\rangle$ inside a cavity through which neutrons and protons pass. For the interaction Hamiltonian, we take

$$H_{int} = g(t)(\sigma^+ a + \sigma^- a^\dagger) \ ,$$

where σ^+ turns a neutron into a proton, σ^- turns a proton into a neutron, and $a^\dagger$, a are meson creation and destruction operators. (The state $|z\rangle$ is an eigenstate of a with eigenvalue z.) Note that H_{int} conserves charge. The coupling $g(t)$ is $1/T$ during the short time T the nucleon passes through the cavity containing the coherent state of mesons; at all other times it is zero. (We neglect the separate Hamiltonians of the nucleon and the mesons for this short time.)

Consider an initial state

$$\frac{1}{\sqrt{2}}(|P\rangle + e^{i\alpha}|N\rangle) \otimes |z\rangle \ ,$$

which contains a coherent superposition of a proton state $|P\rangle$ and a neutron state $|N\rangle$. We can calculate the evolution of this state for large $|z|$. After the nucleon and the meson cavity interact, the state is

$$\frac{1}{\sqrt{2}}\left[\left(e^{i\alpha}\cos\frac{|z|}{\hbar}-i\frac{z}{|z|}\sin\frac{|z|}{\hbar}\right)|N\rangle-\left(ie^{i\alpha}\frac{z^*}{|z|}\sin\frac{|z|}{\hbar}-\cos\frac{|z|}{\hbar}\right)|P\rangle\right]\otimes|z\rangle \quad (11.14)$$

which implies a neutron probability

$$\frac{1}{2}-\frac{z^*e^{i\alpha}-ze^{-i\alpha}}{4i|z|}\sin\frac{2|z|}{\hbar}\ , \quad (11.15)$$

so the relative phase α is observable, relative to the phase defined for the cavity. (See Prob. 11.7.) We could even prepare several meson cavities, which would define different reference phases; to compare the phases, we could send nucleons in the same state through the different cavities.

Our conclusion from this chapter is that, contrary to the claim of Wick, Wightman and Wigner, there are no superselection rules in nonrelativistic quantum mechanics. Any physically meaningful Hermitian operator is measurable, and we find a satisfying agreement between quantum measurement theory and what experiments can measure. (What Hermitian operator is not physically meaningful? An example is the vector potential $\mathbf{A}$. We cannot measure $\mathbf{A}$, not because of a superselection rule, but because $\mathbf{A}$ lacks physical meaning.) This agreement fails when we impose the constraints of relativity on quantum mechanics. Then we find that some Hermitian operators are not observables. This disagreement between quantum measurement theory and what we can actually measure may underlie many unsatisfactory features of quantum field theory. (See Chap. 14.)

Problems

*11.1 Let $\phi(x)$ denote a boson field at a spacetime point x. The sum $\phi(x)+\phi^\dagger(x)$ is a Hermitian operator. In quantum field theory, $\phi(x)$ obeys the following commutation relation:

$$\left[\phi(x)+\phi^\dagger(x),\phi(y)+\phi^\dagger(y)\right]=0, \quad (11.16)$$

whenever the x and y are spacelike separated. Equation (11.16) expresses relativistic causality: since x and y are spacelike separated, the two points cannot be causally related; hence any operator defined at x must commute with any operator defined at y. A fermion field $\psi(x)$, however, obeys an *anti*commutation relation:

$$\left\{\psi(x)+\psi^\dagger(x),\psi(y)+\psi^\dagger(y)\right\}=0\ ,$$

where $\{A,B\}\equiv AB+BA$. Derive a superselection rule for the Hermitian operator $\psi(x)+\psi^\dagger(x)$.

11.2 (a) Derive Eq. (11.5).
(b) Show that if T obeys Eqs. (11.4–5), then T reverses the spin of eigenstates of σ_y if and only if T is antilinear.
(c) Show that T acting on any spin-1/2 state reverses the direction of spin.

11.3 Show that Eq. (11.8) holds.

11.4 Write down the (time-dependent) Schrödinger equation for the Hamiltonian $H_R(t)$ in Eq. (11.10). Assume it has a solution

$$|\psi(t)\rangle = e^{i\varphi(t)} e^{-i(\mathbf{L}+\mathbf{S})\cdot\mathbf{n}\omega t/\hbar}|R\rangle \ ,$$

where $\varphi(t)$ is a phase. (Indeed $|\psi(t)\rangle$ solves the Schrödinger equation to first order in ω.) From $\varphi(t)$ calculate the relative phase between $|R\rangle$ and $|NR\rangle$ after a time $t = 2\pi/\omega$; show that it reduces to π modulo 2π.

11.5 Calculate the tunnelling current between $|NR\rangle$ and $|R\rangle$ (defined in Sect. 11.4) as a function of time if at time $t = 0$ the state is $(|NR\rangle + e^{i\alpha}|R\rangle)/\sqrt{2}$, where α is an arbitrary phase.

11.6 Show how to rewrite $|+\rangle$, defined in Eq. (11.9), as follows:

$$|+\rangle = \frac{1}{2\sqrt{2}} (|1\rangle_{NR} + |0\rangle_{NR}) \otimes (|1\rangle_R + |0\rangle_R) - \frac{1}{2\sqrt{2}} (|1\rangle_{NR} - |0\rangle_{NR}) \otimes (|1\rangle_R - |0\rangle_R) \ .$$

Here $|1\rangle_{NR}$ represents the neutron in the nonrotating box and $|1\rangle_R$ represents the neutron in the rotating box, so these are fermion states; $|0\rangle_{NR}$ and $|0\rangle_R$ represent the boxes empty, so they are boson states. Show that if we remove one of the boxes, the remaining box is in a mixture of two superpositions of fermion and boson states.

*11.7 (a) Show that if $[a, a^\dagger] = 1$ and $a|z\rangle = z|z\rangle$, then $a^\dagger|z\rangle = z^*|z\rangle + |z_\perp\rangle$, where $|z_\perp\rangle$ is normalized and orthogonal to $|z\rangle$.
(b) Find the eigenstates of $\int H_{int} dt = \sigma^+ a + \sigma^- a^\dagger$ in the approximation of large $|z|$, i.e. neglecting the term $|z_\perp\rangle$. (c) Derive Eq. (11.14) in the same approximation. (d) Derive Eq. (11.15).

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[1] G. C. Wick, A. S. Wightman and E. P. Wigner, *Phys. Rev.* **88** (1952) 101.

[2] See, for example, R. P. Feynman, in *Elementary Particles and the Laws of Physics: The 1986 Dirac Memorial Lectures* (Cambridge: Cambridge U. Press), 1987, pp. 1–59.

[3] See Feynman, *op. cit.*; R. F. Streator and A. S. Wightman, *PCT, Spin and Statistics, and All That* (New York: W. A. Benjamin), 1964.

[4] Y. Aharonov and L. Susskind, *Phys. Rev.* **158** (1967) 1237.

[5] Y. Aharonov and L. Susskind, *Phys. Rev.* **155** (1967) 1428.

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12 Quantum Slow Dance

An elegant and useful tool in quantum mechanics is the Born–Oppenheimer approximation (a form of the adiabatic approximation) The Born–Oppenheimer approximation applies to any system with coupled degrees of freedom, when some of them (the "fast" variables) change quickly and all the others (the "slow" variables) change slowly. We treat the motion in two steps. First we "freeze" the slow variables and follow the motion of the fast variables. Anyone speeding on a bicycle past pedestrians understands this step; the pedestrians seem almost stationary. We then compute how the slow variables respond to the average fast motion, just as the pedestrians avoid the passing bicycle. In this approximation, the fast and slow motions couple asymmetrically. The fast motion depends on the configuration of the slow variables; the slow motion does not depend on the configuration of the fast variables, but only on their average motion.

Another elegant and useful tool in quantum mechanics is the Feynman path integral. The Feynman path integral and the adiabatic approximation are related. We begin this chapter with a paradox that leads right to the adiabatic approximation and the Born–Oppenheimer approximation, to Berry's phase and the Feynman path integral.

12.1 A Watched Pot Never Boils

Natura non facit saltum? What Schrödinger called "this damned quantum jumping around" [1] disproves the classical statement that nature never jumps. Consider a simple quantum system: a spin-1/2 particle precessing in a constant magnetic field. The Hamiltonian is

$$H = -\mu B S_z = -\mu B \hbar \sigma_z / 2$$

where the z-axis points along the magnetic field; μ is the magnetic moment. If at time $t = 0$ the state is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left[|\uparrow\rangle + |\downarrow\rangle \right]$$

(where $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of σ_z), at time t it is

$$\begin{aligned} |\psi(t)\rangle &= e^{-iHt/\hbar} |\psi(0)\rangle \\ &= \frac{1}{\sqrt{2}} \left[e^{i\mu Bt/2} |\uparrow\rangle + e^{-i\mu Bt/2} |\downarrow\rangle \right] . \end{aligned} \tag{12.1}$$

At $t = 0$, a measurement of σ_x is sure to yield $\sigma_x = 1$; at time $t = \pi/\mu B$, a measurement is sure to yield $\sigma_x = -1$; at intermediate times, a measurement may yield either result. At

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no time does a measurement of σ_x yield a value other than 1 and -1; the spin jumps from $\sigma_x = 1$ to $\sigma_x = -1$ without passing through any intermediate value. Here is the "damned quantum jumping around" of quantum spin. But nature jumps or not, according to which measurements we make. Equation (12.1) represents a spin precessing with period $2\pi/\mu B$. At any time t, $|\psi(t)\rangle$ is an eigenstate of $\sigma_x \cos(\mu B t) - \sigma_y \sin(\mu B t)$, so repeated measurements of *this* observable do not show any damned jumping.

When does the spin jump? We do not know when it will jump, although we know when it is likely to jump, and that it *must* jump between $t = 0$ and $t = \pi/\mu B$. The probability $P(t)$ that a measurement of σ_x at time t yields the result 1 is

$$P(t) = |\langle\psi(0)|\psi(t)\rangle|^2 = \cos^2(\mu B t/2) \; .$$

A measurement of σ_x at times close to $t = 0$ will likely yield 1, and a measurement of σ_x at times close to $t = \pi/\mu B$ will likely yield -1. The spin flip most likely occurs around $t = \pi/2\mu B$. Can we *observe* when the spin jumps? We can make very many measurements of σ_x between $t = 0$ and $t = \pi/\mu B$ – a dense set of measurements. The jump in σ_x must occur between two successive measurements. When it does, we will know when the jump occurred, to an accuracy Δt equal to the time between the measurements. Since we can make the measurements as dense as we like, we can make Δt as small as we like.

But now we seem to contradict the uncertainty relation for energy and time. Section 8.5 derives the uncertainty relation

$$\Delta E \Delta t \geq \hbar/2$$

with the interpretation that ΔE is the uncertainty in the energy of a system and Δt is the uncertainty in the time *defined by the system*. Here, the spin defines a moment in time by jumping. We measure the time of the jump to an accuracy Δt. Over a period of time Δt the energy is uncertain because we don't know when the spin flips. But ΔE cannot be larger than $\mu\hbar B$, the energy gap between the states $|\uparrow\rangle$ and $|\downarrow\rangle$. Hence we cannot make Δt arbitrarily small. Yet we can make the measurements of σ_x as dense as we like!

So we arrive at a paradox. The spin must flip, because (by assumption) at some time we find the spin flipped. We can certainly make measurements of σ_x as dense as we like, and thus measure the time of the jump in σ_x as precisely as we like. Yet the uncertainty principle states that we *cannot* measure the time of the jump as precisely as we like. This paradox is the quantum Zeno paradox [2], named after the ancient Greek philosopher who tried to understand motion by dividing time into shorter and shorter intervals, and found himself proving that motion is impossible. The quantum Zeno paradox arises whether we consider a spin, an excited atom decaying to its ground state, or a particle tunnelling through a potential barrier. The question is always the same: Can we find out when the system changes by continuously observing it?

12.2 The Adiabatic Approximation

There are no quantum jumps in this section. With no quantum jumps, this section may seem unconnected with the last. Actually, it is closely related to the quantum Zeno paradox, but the relation is explicit only in the next section. In this section, we eliminate quantum jumps.

How do we eliminate quantum jumping? Consider a closed system in an eigenstate of H_f, a Hamiltonian with discrete, nondegenerate eigenvalues. If H_f does not depend on time, the system never jumps to another state. What if H_f does depend on time? It *can* depend on time. If we prepare the system in an eigenstate of H_f, and H_f changes quickly, the system may jump to another state. But let H_f change adiabatically (slowly); if H_f changes slowly enough, the system never jumps to another state. Instead of jumping, it adjusts itself to the changing Hamiltonian. The system behaves like a heavy weight hanging on a thin string. Pull the string quickly – it snaps and the weight falls. Pull the string slowly – the weight comes up with it.

"Slowly enough" has the following formal sense. Let H_f depend on time only through a changing vector $\mathbf{R}$. The components of $\mathbf{R}$ could be quantum observables, but for now let us treat them as slowly changing (classical) parameters. The eigenvalues and eigenstates of H depend on $\mathbf{R}$, so we write $H(\mathbf{R})|\Psi_i(\mathbf{R})\rangle = E_i(\mathbf{R})|\Psi_i(\mathbf{R})\rangle$. Let a function $\mathbf{r}(t/T)$, for $0 \le t \le T$, define a path in the space of parameters, and let $\mathbf{R}[t] \equiv \mathbf{r}[t/T]$. That is, $\mathbf{R}[t]$ evolves from the beginning to the end of the path over the time interval $0 \le t \le T$; the larger T, the slower the evolution. At time $t = 0$ we prepare the system in the state $|\Psi_i(\mathbf{R}[0])\rangle$. According to the quantum adiabatic theorem [3], the state of the system at time $t = T$ is $|\Psi_i(\mathbf{R}[T])\rangle$, with probability tending to 1 as T tends to infinity. At time t the state of the system is $|\Psi_i(\mathbf{R}[t])\rangle$.

The system also acquires a phase $\phi_i(t)$. Thus the state of the system at time t, in the adiabatic approximation, is

$$e^{i\phi_i(t)}|\Psi_i(\mathbf{R}[t])\rangle \ . \tag{12.2}$$

To determine $\phi_i(t)$, we substitute Eq. (12.2) into the time-dependent Schrödinger equation and take the inner product of both sides with Eq. (12.2):

$$\frac{d}{dt}\phi_i(t) = i\langle\Psi_i(\mathbf{R})|\nabla_{\mathbf{R}}|\Psi_i(\mathbf{R})\rangle \cdot \frac{d\mathbf{R}}{dt} - \frac{1}{\hbar}E_i(\mathbf{R}) \ .$$

Thus

$$\begin{aligned}\phi_i(t) - \phi_i(0) &= \int_0^t dt' \left[i\langle\Psi_i(\mathbf{R})|\nabla_{\mathbf{R}}|\Psi_i(\mathbf{R})\rangle \cdot \frac{d\mathbf{R}}{dt'} - \frac{1}{\hbar}E_i(\mathbf{R}) \right] \\ &= \int_{\mathbf{R}[0]}^{\mathbf{R}[t]} \langle\Psi_i(\mathbf{R})|i\nabla_{\mathbf{R}}|\Psi_i(\mathbf{R})\rangle \cdot d\mathbf{R} - \frac{1}{\hbar}\int_0^t dt' \ E_i(\mathbf{R}) \ . \end{aligned} \tag{12.3}$$

The integrand $\mathbf{A}_B \equiv \langle\Psi_i(\mathbf{R})|i\nabla_{\mathbf{R}}|\Psi_i(\mathbf{R})\rangle$ is *Berry's connection* [4] for the state $|\Psi_i(\mathbf{R})\rangle$. The integral $-\int_0^t E_i dt'/\hbar$ is called the *dynamical* phase.

Why compute the phase? The overall phase of a quantum state has no physical meaning. But a quantum system may be in a superposition of states, each with its own phase. The *relative* phase of these states is observable. Consider two paths $\mathbf{R}[t]$ and $\mathbf{R}'[t]$ with the same endpoints $\mathbf{R}[0] = \mathbf{R}'[0]$ and $\mathbf{R}[T] = \mathbf{R}'[T]$, and suppose that the system evolves in a superposition of states $|\Psi(\mathbf{R}[t])\rangle$ and $|\Psi(\mathbf{R}'[t])\rangle$. At time $t = T$, the relative phase of this superposition contains two parts. One part is the relative dynamical phase (which vanishes if the states acquire the same dynamical phase). The other part of the relative phase, *Berry's phase*, is the difference between $\mathbf{A}_B$ integrated along $\mathbf{R}$ and $\mathbf{A}_B$ integrated along $\mathbf{R}'$, i.e. it is the circular integral of $\mathbf{A}_B$ along the *closed* path comprising $\mathbf{R}$ and $\mathbf{R}'$ with opposite senses.

The relative phase is observable, hence Berry's phase must be well defined. Let's check – is the circular integral

$$\oint \mathbf{A}_B \cdot d\mathbf{R}$$

well defined? Is the connection $\mathbf{A}_B = \langle \Psi_i(\mathbf{R}) | i\nabla_{\mathbf{R}} | \Psi_i(\mathbf{R}) \rangle$ well defined? Actually, $\mathbf{A}_B$ is somewhat arbitrary, because $|\Psi_i(\mathbf{R})\rangle$ is somewhat arbitrary: if we multiply $|\Psi_i(\mathbf{R})\rangle$ by a phase factor $e^{i\Lambda(\mathbf{R})}$, it remains the same instantaneous eigenstate of $H_f(\mathbf{R})$; but $\mathbf{A}_B$ changes by $-\nabla_{\mathbf{R}}\Lambda(\mathbf{R})$. If $\mathbf{A}_B$ is not well defined, how can its integral, Berry's phase, be well defined?

This question has an elegant answer. Multiplying $|\Psi_i(\mathbf{R})\rangle$ by the phase factor $e^{i\Lambda(\mathbf{R})}$ is a local phase transformation – local in $\mathbf{R}$-space. It is analogous to the local phase transformation (in x-space) of Eq. (4.3). And $\mathbf{A}_B$ is analogous to the electromagnetic vector potential $\mathbf{A}$. Since a local *phase* transformation of $|\Psi_i(\mathbf{R})\rangle$ induces a *gauge* transformation of $\mathbf{A}_B$, these two transformations go hand in hand, just as they do in Sect. 4.3. Now Berry's phase is the integral of $\mathbf{A}_B$ along a closed loop; so it is gauge invariant and well defined, just as the integral of the electromagnetic vector potential along a closed loop is gauge invariant and well defined and equal to the magnetic flux through the loop.

The thought experiment of Sect. 11.4 illustrates Berry's phase. A magnetic field slowly rotates, and a neutron polarized along the magnetic field rotates with it. For each complete rotation, the neutron acquires a phase π (Berry's phase) in addition to its dynamical phase. This phase is observable.

Berry's connection and Berry's phase appear in several contexts in this chapter. In this section, the context is the Born–Oppenheimer approximation [5]. In the Born–Oppenheimer approximation, the components of $\mathbf{R}$ are not classical; they are quantum observables and may not even commute. They evolve according to their own "slow" Hamiltonian H_s, and the overall Hamiltonian is the sum $H = H_f + H_s$. By itself, the "fast" Hamiltonian H_f does not induce quantum jumps; it does not change $\mathbf{R}$, and its eigenvalues are (by assumption) discrete and nondegenerate. The slow Hamiltonian changes $\mathbf{R}$ and can induce quantum jumps, which complicate the overall evolution. Yet there is still a limit – the adiabatic limit – in which the evolution simplifies. In the adiabatic limit, we treat H_s as an arbitrarily weak perturbation on H_f. The weaker the perturbation, the smaller the probability of transitions (quantum jumps) among the eigenstates of H_f. The unperturbed Hilbert space for H divides into subspaces, one for each eigenvalue E_i of H_f. In the adiabatic limit, H_s does not cause transitions among the subspaces; the "fast" variables remain in an eigenstate $|\Psi_i(\mathbf{R})\rangle$ of H_f, with i fixed, while dynamical and Berry phases of $|\Psi_i(\mathbf{R})\rangle$ show up in H as induced scalar and vector potentials.

Let $\mathbf{\Pi}_i$ denote the projector onto the subspace corresponding to E_i. These subspaces are disjoint and form a complete set; hence [6]

$$\sum_i \mathbf{\Pi}_i = 1 \ .$$

If the adiabatic limit is a good approximation, we can replace H_s by $\sum_i \mathbf{\Pi}_i H_s \mathbf{\Pi}_i$ (no quantum jumps) to obtain the effective Hamiltonian of Born and Oppenheimer:

$$H_{eff} = H_f + \sum_i \mathbf{\Pi}_i H_s \mathbf{\Pi}_i \ .$$

In H_{eff} there are induced potentials. To exhibit them, we consider [7] a generic Hamiltonian for $\mathbf{R}$ and its conjugate momentum $\mathbf{P}$:

$$H_s = \frac{P^2}{2M} + V(\mathbf{R}) \ ,$$

where M is large. The sum $\sum_i \mathbf{\Pi}_i H_s \mathbf{\Pi}_i$ in H_{eff} contains products of the form

$$\mathbf{\Pi}_i P^2 \mathbf{\Pi}_i = \sum_j \mathbf{\Pi}_i \mathbf{P} \mathbf{\Pi}_j \mathbf{P} \mathbf{\Pi}_i \ . \tag{12.4}$$

We can simplify Eq. (12.4) if we decompose $\mathbf{P}$ into two parts, $\mathbf{P} = (\mathbf{P} - \mathbf{A}) + \mathbf{A}$. The first part acts only *within* each subspace; that is,

$$[\mathbf{P} - \mathbf{A}, \mathbf{\Pi}_i] = 0 \ , \tag{12.5}$$

for every i. Only the second part, $\mathbf{A}$, causes transitions among the subspaces. However, $\mathbf{A}$ (like a vector potential) is somewhat arbitrary: if we add to $\mathbf{A}$ any term that commutes with the $\mathbf{\Pi}_i$, then $\mathbf{A}$ still satisfies Eq. (12.5). We remove this arbitrariness by requiring

$$\mathbf{\Pi}_i \mathbf{A} \mathbf{\Pi}_i = 0 \tag{12.6}$$

for each i. Equations (12.5–6) define $\mathbf{A}$ completely. In fact,

$$\mathbf{A} = \mathbf{P} - \sum_i \mathbf{\Pi}_i \mathbf{P} \mathbf{\Pi}_i = \frac{1}{2} \sum_i [\mathbf{\Pi}_i, [\mathbf{\Pi}_i, \mathbf{P}]] \ . \tag{12.7}$$

Applying Eqs. (12.5–6) to Eq. (12.4), we get

$$\begin{aligned} \mathbf{\Pi}_i P^2 \mathbf{\Pi}_i &= \sum_j \mathbf{\Pi}_i (\mathbf{P} - \mathbf{A} + \mathbf{A}) \mathbf{\Pi}_j (\mathbf{P} - \mathbf{A} + \mathbf{A}) \mathbf{\Pi}_i \\ &= (\mathbf{P} - \mathbf{A})^2 \mathbf{\Pi}_i + \mathbf{\Pi}_i \mathbf{A}^2 \mathbf{\Pi}_i \ . \end{aligned}$$

The effective Hamiltonian for the $\mathbf{R}$, then, is

$$H_{eff} = H_f + \frac{1}{2M} (\mathbf{P} - \mathbf{A})^2 + \frac{1}{2M} \sum_i \mathbf{\Pi}_i \mathbf{A}^2 \mathbf{\Pi}_i + V(\mathbf{R}) \ . \tag{12.8}$$

The sum in i is an induced scalar potential, while $\mathbf{A}$ is an induced vector potential.[1] Indeed, $\mathbf{A}$ is Berry's connection $\mathbf{A}_B$ in a non-abelian gauge.

As an example of an effective Hamiltonian, consider a very long, straight solenoid with all its mass M concentrated at one end. The massive end moves freely, with coordinate $\mathbf{R}$, while the rest of the solenoid slews around in such a way as to always include the point $\mathbf{R} = \mathbf{0}$. The magnetic field inside the solenoid is $B\mathbf{R}/R$. At $\mathbf{R} = \mathbf{0}$ sits a spin-$1/2$ particle with fast Hamiltonian

$$H_f = \frac{\mu B \hbar}{2R} \mathbf{R} \cdot \sigma \ , \tag{12.9}$$

[1] Part or all of the difference $P^2/2M - \sum_i \mathbf{\Pi}_i P^2 \mathbf{\Pi}_i / 2M$ is proportional to the velocity $(\mathbf{P} - \mathbf{A})/M$ of the "slow" variables. Hence the slow variables *must* be slow, for the Born–Oppenheimer approximation to be applicable; the limit $M \to \infty$ is not sufficient.

where μ is constant. H_f induces two subspaces, with projectors

$$\mathbf{\Pi}_\pm = \frac{1 \pm \mathbf{R}\cdot\sigma/R}{2} \ . \tag{12.10}$$

Inserting Eq. (12.10) into Eq. (12.7), we obtain

$$\mathbf{A} = [\mathbf{R}\cdot\sigma/2R, [\mathbf{R}\cdot\sigma/2R, \mathbf{P}]] = \frac{\hbar\mathbf{R}\times\sigma}{2R^2} \tag{12.11}$$

as the induced vector potential for this problem. This $\mathbf{A}$ is guaranteed to satisfy the two conditions (12.5–6). If we consider the subspace defined by $\mathbf{\Pi}_+$, the effective Hamiltonian for the massive end of the solenoid is

$$H_{eff} = \frac{\mu B\hbar}{2} + \frac{1}{2M}(\mathbf{P} - \frac{\hbar\mathbf{R}\times\sigma}{2R^2})^2 + \frac{\hbar^2}{4MR^2} + V(\mathbf{R}) \ .$$

Thus the mass moves in a scalar potential $\mu B\hbar/2 + \hbar^2/4MR^2 + V(\mathbf{R})$ and an $SU(2)$ vector potential for an effective field

$$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = \frac{1}{2}\epsilon_{ijk}(\partial_j A_k - \partial_k A_j - i[A_j, A_k]) = -\frac{\hbar R_i}{2R^4}(\mathbf{R}\cdot\sigma) \ .$$

(See Sect. 4.8 and Prob. 4.10.) Since $\sigma\cdot\mathbf{R}/R = 1$ in this subspace, we have $\mathbf{B} = -\hbar\mathbf{R}/2R^3$. The solution to Eqs. (12.5–6) yields an effective magnetic monopole at $\mathbf{R} = \mathbf{0}$ – the mass at $\mathbf{R}$ behaves like a charge in the field of a monopole at the origin! The induced field $\mathbf{B}$ is abelian, although the abelian monopole appears in a non-abelian representation: it has no string.

We can generalize this result to arbitrary spin; if we write Eqs. (12.5–6) as algebraic equations, the result holds for any representation of the algebra. So instead of Eq. (12.5) we write

$$[\mathbf{P} - \mathbf{A}, H_f] = 0 \ . \tag{12.5$'$}$$

Equation (12.5′) is equivalent to Eq. (12.5) if the eigenvalues of H_f are independent of $\mathbf{R}$, as they are in Eq. (12.9). Instead of Eq. (12.6) we write

$$\mathbf{A} = [H_f, \mathbf{\Lambda}] \ , \tag{12.6$'$}$$

for some $\mathbf{\Lambda}$. Equation (12.6′) implies Eq. (12.6) because if $\mathbf{A}$ satisfies Eq. (12.6′) then, for every i,

$$\begin{aligned}\mathbf{\Pi}_i\mathbf{A}\mathbf{\Pi}_i &= \mathbf{\Pi}_i[H_f, \mathbf{\Lambda}]\mathbf{\Pi}_i \\ &= \mathbf{\Pi}_i\,[\sum_j \mathbf{\Pi}_j H_f \mathbf{\Pi}_j, \mathbf{\Lambda}]\,\mathbf{\Pi}_i = 0 \ .\end{aligned}$$

For $H_f = \mu B\mathbf{R}\cdot\mathbf{S}/R$ we have $\mathbf{A} = [H_f, i\mathbf{S}/\mu B\hbar R] = \mathbf{R}\times\mathbf{S}/R^2$ as the solution to Eqs. (12.5′-6′), thus the induced field is $\mathbf{B} = -(\mathbf{R}\cdot\mathbf{S})\mathbf{R}/R^4$ for arbitrary spin $\mathbf{S}$.

So far we have assumed the eigenvalues of H_f to be discrete and nondegenerate. If H_f has a discrete and *degenerate* eigenvalue, Berry's phase (and the induced field) may be non-abelian

[8]. The eigenstates belonging to this eigenvalue do not (in the adiabatic approximation) jump to eigenstates belonging to other eigenvalues, but they may mix among themselves. The mixing amounts to multiplication by a non-abelian phase (i.e., a unitary matrix). Consider, for example, the Hamiltonian

$$H = \frac{P^2}{2M} + V(\mathbf{R}) + g(\mathbf{R}\cdot\mathbf{S}/R)^2 \ ,$$

with $\mathbf{S}$ representing a spin-$3/2$ particle. The fast Hamiltonian $H_f = g(\mathbf{R}\cdot\mathbf{S}/R)^2$ has two degenerate eigenvalues, $g\hbar^2/4$ and $9g\hbar^2/4$. What potential $\mathbf{A}$ satisfies the equations for the vector potential? Applying Eq. (12.7) we find

$$\begin{aligned}\mathbf{A} &= \frac{1}{4\hbar^2}[\ (\mathbf{R}\cdot\mathbf{S}/R)^2,\ [(\mathbf{R}\cdot\mathbf{S}/R)^2, \mathbf{P}]\ ] \\ &= \frac{1}{4\hbar^2 R^2}\left[4(\mathbf{R}\cdot\mathbf{S}/R)(\mathbf{R}\times\mathbf{S})(\mathbf{R}\cdot\mathbf{S}/R) + \hbar^2 R^2 \mathbf{R}\times\mathbf{S}\right] \ .\end{aligned}$$

The induced field is

$$\mathbf{B} = \frac{(\mathbf{R}\cdot\mathbf{S})\mathbf{R}}{2\hbar^2 R^4}[2\mathbf{S}^2 - \frac{\hbar^2}{2} - \frac{4}{R^2}(\mathbf{R}\cdot\mathbf{S})^2] \ .$$

For the states with $\mathbf{R}\cdot\mathbf{S}/R = \pm 3\hbar/2$, $\mathbf{B}$ reduces to $-(\mathbf{R}\cdot\mathbf{S})\mathbf{R}/R^4$ as in the abelian example; it just happens that these two degenerate states do not mix. For the states with $\mathbf{R}\cdot\mathbf{S}/R = \pm\hbar/2$, however, we find $\mathbf{B} = 3(\mathbf{R}\cdot\mathbf{S})\mathbf{R}/R^4$, which is -3 times the abelian example; for these states $\mathbf{B}$ contains a truly non-abelian contribution.

12.3 Feynman Paths

Alongside the Schrödinger and Heisenberg formulations of quantum mechanics is the Feynman *path integral* formulation [9]. Feynman considered all possible paths of a particle from an initial spacetime point $(\mathbf{x}_i, t_i)$ to a final spacetime point $(\mathbf{x}_f, t_f)$. To each path $\mathbf{x}(t)$ he assigned a phase

$$e^{i\int_{t_i}^{t_f} L dt/\hbar} \ ,$$

where L is the classical Lagrangian evaluated along the path. Feynman's path integral is a normalized sum of the phases of all these paths, and equals the value of the wave function of the particle at the point $\mathbf{x}_f$ at time t_f, if at time t_i the particle was at $\mathbf{x}_i$.

We may object that quantum particles do not move along paths. A path that defines the position and velocity of a particle at each moment violates the uncertainty principle. How can the path integral be consistent with the "quantum jumping around" of an electron in an atom? To answer this objection, let us return to the quantum Zeno paradox of Sect. 12.1. Once we resolve the paradox, we will appreciate the physical meaning of a Feynman path, and will see how the path integral formulation follows from the formulations of Schrödinger and Heisenberg.

A direct way to resolve the quantum Zeno paradox is to calculate what happens. The state of the spin is $|\psi(0)\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ at time $t = 0$. Suppose we measure σ_x a total of N

times, at equal time intervals, over a period of time T. The interval between measurements is T/N. What is the probability of finding the spin unchanged after the first measurement? According to Eq. (12.1), the state at time $t = T/N$ is

$$\frac{1}{\sqrt{2}}\left[e^{i\mu BT/2N}|\uparrow\rangle + e^{-i\mu BT/2N}|\downarrow\rangle\right] ,$$

so the probability of finding the spin unchanged is $\cos^2(\mu BT/2N)$. Hence the probability of finding the spin unchanged at time T, after N measurements, is $\cos^{2N}(\mu BT/2N)$. As N approaches infinity, $\cos^{2N}(\mu BT/2N)$ approaches 1. (See Prob. 12.8.) So the spin never flips! In particular, consider the spin between the times $t = 0$ and $t = T = \pi/\mu B$. If we measure σ_x *only* at time $t = T$, we will certainly find the spin flipped; if we measure σ_x densely between $t = 0$ and $t = T$, we will not find the spin flipped at any time. If we continuously observe a quantum jump, it never occurs. A watched pot never boils.

Shades of Zeno! We have proved that motion is impossible. But motion is not *always* impossible. At any time t, the state $|\psi(t)\rangle$ in Eq. (12.1) is an eigenstate of $\sigma_x \cos(\mu Bt) - \sigma_y \sin(\mu Bt)$, so repeated measurements of *this* operator shows the spin to be precessing as in Eq. (12.1). Motion is possible. Yet the resolution of the quantum Zeno paradox is a surprise. Consider an experiment that is dual to the previous one, in the following sense: instead of continuously observing σ_x on a spin that is precessing, we continuously observe $\sigma_x \cos(\mu Bt) - \sigma_y \sin(\mu Bt)$ on a spin that is *not* precessing. The Hamiltonian vanishes; there is no magnetic field. But now we measure $\sigma_x \cos(\mu Bt) - \sigma_y \sin(\mu Bt)$ at times $t = n\pi/\mu BN$, where $n = 1, 2, 3, \ldots, N$. In the limit $N \to \infty$, we find the spin precessing just as it did before; each measurement of $\sigma_x \cos(\mu Bt) - \sigma_y \sin(\mu Bt)$ yields 1. Here continuous observation – rather than a magnetic field – causes the spin to precess. A watched pot boils – without a stove!

We can continuously observe any quantum state. If we continuously observe a decaying atom, it never decays; if we continuously check whether a particle has crossed a barrier, it never crosses. We can make a free particle scatter off a force-free region just by constantly checking whether the particle has entered the region. Indeed, we can induce any evolution we want in a quantum system.

Here is a general proof that continuous observation dictates the evolution of a quantum system. Let $|\Psi(\mathbf{R})\rangle$, the state of a quantum system, depend parametrically on a vector $\mathbf{R}$; $\mathbf{R}$ could represent e.g. the polarization direction of a spin as in Prob. 12.4, or the displacement of a localized state as in Prob. 12.7. The vector $\mathbf{R}$ changes in time along a smooth path $\mathbf{R}[t]$. We check whether the system evolves along the path $\mathbf{R}[t]$ by making a dense set of measurements. Define $t_n \equiv nT/N$, where $n = 1, \ldots, N$. At times $t = t_n$ we check whether the system is in the state $|\Psi(\mathbf{R}[t_n])\rangle$. Assuming we find it in the state $|\Psi(\mathbf{R}[t_n])\rangle$ at time $t = t_n$, what is the probability we will find the system in the state $|\Psi(\mathbf{R}[t_{n+1}])\rangle$ at time $t = t_{n+1}$? It is $|K(t_{n+1}, t_n)|^2$, where

$$K(t_{n+1}, t_n) \equiv \langle\Psi(\mathbf{R}[t_{n+1}])|e^{-iHT/N\hbar}|\Psi(\mathbf{R}[t_n])\rangle$$

and H is the Hamiltonian of the system. Now we evaluate $K(t_{n+1}, t_n)$ to order T/N. To this order, $K(t_{n+1}, t_n)$ reduces to two factors. One is a phase factor,

$$e^{-i\langle H\rangle T/N\hbar} , \tag{12.12}$$

arising from the Hamiltonian: $\langle H\rangle$ is the expectation value of H in the state $|\Psi(\mathbf{R}[t_n])\rangle$. The second factor in $K(t_{n+1},t_n)$ is

$$\langle\Psi(\mathbf{R}[t_{n+1}])|\Psi(\mathbf{R}[t_n])\rangle\ . \tag{12.13}$$

To order T/N, it, too, is a phase factor. We can write it

$$1-\frac{T}{N}\langle\Psi(\mathbf{R}[t_n])|\frac{d}{dt_n}|\Psi(\mathbf{R}[t_n])\rangle\ . \tag{12.14}$$

Since $K(t_{n+1},t_n)$ reduces to a phase factor, to order T/N, the probability $|K(t_{n+1},t_n)|^2$ equals 1 to order T/N. It then follows from Prob. 12.8 that the probability for the system to be in the state $|\Psi(R[t_n])\rangle$ at time $t=t_n$ *for all* n approaches 1 as N approaches infinity – the limit of continuous observation. The quantum world fulfills our every expectation; it confirms all our hopes, and also all our fears. Hence, be an optimist!

At time $t=t_N=T$, we find the system in the state $|\Psi(\mathbf{R}[T])\rangle$, up to a phase factor. An overall phase is not measurable. All the same, we will compute it; we will discover in it the *relative* phase of a Feynman path, and then derive the path integral. The phase is the sum of the phases in Eq. (12.12) and Eq. (12.14) for all n. The product over all n of Eq. (12.12) approaches $e^{i\Phi_D}$, where the dynamical phase Φ_D is

$$\Phi_D=-\int_0^T\langle\Psi(\mathbf{R}[t])|H|\Psi(\mathbf{R}[t])\rangle dt/\hbar\ ; \tag{12.15}$$

Φ_D generalizes the dynamical phase of Sect. 12.2. (In Sect. 12.2, $|\Psi(\mathbf{R})\rangle$ is an eigenstate of the Hamiltonian with eigenvalue E_i and Φ_D reduces to $-\int_0^T E_i dt/\hbar$.) The product over all n of Eq. (12.14) approaches $e^{i\Phi_B}$, where Φ_B is the integral of Berry's connection:

$$\Phi_B=\int_{\mathbf{R}[0]}^{\mathbf{R}[T]}\langle\Psi(\mathbf{R})|i\nabla_{\mathbf{R}}|\Psi(\mathbf{R})\rangle\cdot d\mathbf{R}\ . \tag{12.16}$$

Although this integral is not invariant under a change in the phase of $|\Psi(\mathbf{R})\rangle$, the path integral depends only on relative phases among paths, and the relative phase is invariant. The evolution of $|\Psi(\mathbf{R})\rangle$ is not adiabatic here – $|\Psi(\mathbf{R})\rangle$ evolves according to continuous observation – but we can still define (and measure) Φ_B for any $\mathbf{R}[t]$ that defines a closed path in Hilbert space [10]. (See Prob. 12.10.)

Let us now show that the sum of phases $\Phi_B+\Phi_D$ equals the time integral of the Lagrangian of the system, divided by $\hbar$. We assume a generic Hamiltonian $H=p^2/2m+V(\mathbf{x})$ for a three-dimensional particle. For $|\Psi(\mathbf{R})\rangle$ we take the normalized wave packet

$$|\mathbf{X},\mathbf{P}\rangle=(\pi\Delta^2)^{-3/4}e^{-(\mathbf{x}-\mathbf{X})^2/2\Delta^2}e^{i\mathbf{P}\cdot\mathbf{x}/\hbar}\ , \tag{12.17}$$

a coherent state in the six-dimensional phase space of the particle. The parameters $\mathbf{X}$, $\mathbf{P}$ represent displacements in phase space. Consider a smooth path $\mathbf{X}[t],\mathbf{P}[t]$ in phase space and a sequence of measurements to check whether the particle is in the state $|\mathbf{X}[t],\mathbf{P}[t]\rangle$ at time t. As the measurements become denser (the limit of continuous observation), the probability to

find the particle evolving along this path approaches 1. What is its phase? To obtain Φ_D we compute

$$\begin{aligned}\langle \mathbf{X},\mathbf{P}|H|\mathbf{X},\mathbf{P}\rangle &= \langle \mathbf{X},\mathbf{P}|[\frac{p^2}{2m}+V(\mathbf{x})]|\mathbf{X},\mathbf{P}\rangle \\ &= \langle 0,0|[\frac{(\mathbf{p}+\mathbf{P})^2}{2m}+V(\mathbf{x}+\mathbf{X})]|0,0\rangle \\ &= \frac{P^2}{2m}+V(\mathbf{X})+\mathcal{O}(\Delta^2)+\langle 0,0|\frac{p^2}{2m}|0,0\rangle\ . \end{aligned} \tag{12.18}$$

The term $\langle 0,0|(p^2/2m)|0,0\rangle$ in Eq. (12.18) is the same for all the paths and we drop it, integrating the other terms to obtain Φ_D. To obtain Φ_B we compute the Berry connection:

$$\begin{aligned}&\langle \mathbf{X},\mathbf{P}|i\nabla_{\mathbf{X}}|\mathbf{X},\mathbf{P}\rangle = 0\ , \\ &\langle \mathbf{X},\mathbf{P}|i\nabla_{\mathbf{P}}|\mathbf{X},\mathbf{P}\rangle = -\mathbf{X}/\hbar\ . \end{aligned} \tag{12.19}$$

Integrating the connection by parts we have $\Phi_B = -\int \mathbf{X}\cdot d\mathbf{P}/\hbar = \int \mathbf{P}\cdot d\mathbf{X}/\hbar$ up to end terms that are the same for all paths. Combining Φ_D and Φ_B, we find that the phase of a path in $\mathbf{X},\mathbf{P}$ approaches [11]

$$\Phi_B+\Phi_D = \int [\mathbf{P}\cdot\dot{\mathbf{X}}-\frac{P^2}{2m}-V(\mathbf{X})+\mathcal{O}(\Delta^2)]dt/\hbar+[\mathbf{X}(0)\mathbf{P}(0)-\mathbf{X}(T)\mathbf{P}(T)]/\hbar\ . \tag{12.20}$$

In classical mechanics, $\mathbf{P}\cdot\dot{\mathbf{X}}-H$ and L are identical; hence Eq. (12.20) contains the phase $\int L dt/\hbar$. Indeed, the relative phase $\Phi_B+\Phi_D$ always reduces to an integral $\int L dt/\hbar$ for some Lagrangian L. (See Prob. 12.12.)

We now apply a mathematical trick:

$$\frac{1}{(2\pi\hbar)^3}\int_{-\infty}^{\infty} d^3\mathbf{X}d^3\mathbf{P}\ \ |\mathbf{X},\mathbf{P}\rangle\langle\mathbf{X},\mathbf{P}| = 1\ . \tag{12.21}$$

The set of states $|\mathbf{X},\mathbf{P}\rangle$ is *overcomplete*, but Eq. (12.21) is the identity operator, so we can introduce it at times $t_1,t_2,\ldots,t_{N-1}$ into

$$K(T,0) = \langle \mathbf{X}(T),\mathbf{P}(T)|e^{-iHT/\hbar}|\mathbf{X}(0),\mathbf{P}(0)\rangle$$

to get (up to an overall phase)

$$\begin{aligned}K(T,0) &= \int_{-\infty}^{\infty}\left[\prod_{n=1}^{N}\langle \mathbf{X}_n,\mathbf{P}_n|e^{-iHT/N\hbar}|\mathbf{X}_{n-1},\mathbf{P}_{n-1}\rangle\right]\prod_{n=1}^{N-1}\frac{d^3\mathbf{X}_n d^3\mathbf{P}_n}{(2\pi\hbar)^3} \\ &= \int_{-\infty}^{\infty} e^{iT\sum_{n=1}^{N}[\mathbf{P}_n\cdot\dot{\mathbf{X}}_n-P_n^2/2m-V(\mathbf{X}_n)+\mathcal{O}(\Delta^2)]/N\hbar}\prod_{n=1}^{N-1}\frac{d^3\mathbf{X}_n d^3\mathbf{P}_n}{(2\pi\hbar)^3}\ , \end{aligned} \tag{12.22}$$

where we define $\mathbf{X}_n\equiv\mathbf{X}(t_n)$, $\mathbf{P}_n\equiv\mathbf{P}(t_n)$ for $n=0,1,\ldots,N$, and

$$\dot{\mathbf{X}}_n \equiv \frac{\mathbf{X}_n-\mathbf{X}_{n-1}}{T/N}\ .$$

Equation (12.22) is a path integral in phase space. We obtain Feynman's path integral in configuration space (i.e. the sum of all paths from initial position $\mathbf{X}(0)$ to final position $\mathbf{X}(T)$) in two steps. First, we take the limit $\Delta \to 0$. In this limit, $|\mathbf{X}_n, \mathbf{P}_n\rangle$ reduces[2] to a δ-function of $\mathbf{X}_n$. Second, we integrate over $d^3\mathbf{P}_1, \ldots, d^3\mathbf{P}_{N-1}$, to obtain (up to normalization)

$$\int_{-\infty}^{\infty} d^3\mathbf{X}_1 \ldots d^3\mathbf{X}_{N-1} e^{iT\sum_{n=1}^{N}\left[m(\dot{\mathbf{X}}_n)^2/2 - V(\mathbf{X}_n)\right]/N\hbar} . \tag{12.23}$$

As N tends to infinity, the paths in the Feynman path integral become infinitely kinky; only a subset of the paths, of measure zero, are smooth as we have assumed in deriving Eq. (12.20). Yet Eq. (12.23) remains valid in the limit $N \to \infty$. (See Prob. 12.14.) The square of the absolute value of Eq. (12.23) is the probability density for a particle at $\mathbf{X}_0$ at time $t = 0$ to be at $\mathbf{X}_N$ at time $t = T$.

12.4 Classical Analogues

Section 12.2 presents Berry's phase as a quantum effect, which, of course, it is. But Berry's phase has a classical analogue: Hannay's angle [12]. (See Prob. 12.15.) It is not surprising that Berry's phase, which shows up in the classical action of a path integral, has a classical analogue. The Born–Oppenheimer approximation, too, has a classical analogue. But some quantum effects do not have classical analogues. (See Chap. 6.) What does it mean for a quantum effect to have – or not have – a classical analogue?

Both Berry's phase and the Aharonov–Bohm effect can affect electron interference. Consider an example of a two-slit experiment in which polarized electrons pass through a magnetic field, constant in magnitude but not in direction. (See Fig. 12.1.) Suppose that the electrons are initially polarized along the magnetic field. Then, if the change in the direction of the magnetic field is sufficiently gradual, the electrons will remain polarized along the field, and rotate with it. If the amount of rotation depends on the path of the electron, a Berry phase will appear as a

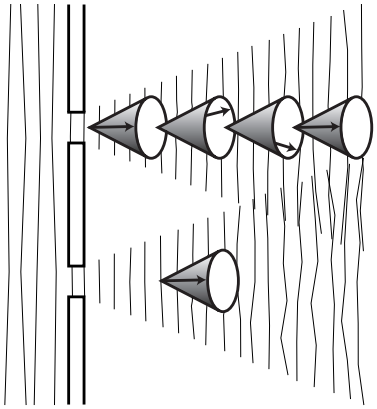

Figure 12.1: A two-slit interference experiment with a magnetic field. The magnitude of the magnetic field is constant, but its direction is not: in the region of the lower wave packet the direction is constant, while in the region of the upper wave packet the direction gradually sweeps out a cone, in such a way that the initial and final directions in the upper region coincide with the direction in the lower region.

[2] The reduction leaves behind the end terms $[\mathbf{X}_0\mathbf{P}_0 - \mathbf{X}_N\mathbf{P}_N]/\hbar$ that appear in Eq. (12.20); they are arbitrary in the limit $\Delta \to 0$ and we drop them.

shift in the pattern of interference between the two paths, just as if the paths went either side of a shielded solenoid. But the Berry phase has a classical analogue; the Aharonov–Bohm effect does not.

There is another difference between the Berry phase and the Aharonov–Bohm effect. The Berry phase, unlike the Aharonov–Bohm effect, arises locally. Suppose that many physicists probe the region where the electrons go – and only this region; each physicist, working alone, checks a small, simply connected part of the region. The physicists then organize a conference and compare their results. If they measured the magnetic field everywhere in the region where the electrons pass, they can predict the shift in the electron interference due to Berry's phase. They do not have to do the interference experiment. But whatever they measure, they cannot predict the shift due to the Aharonov–Bohm effect; the vector potential is not measurable. They have to do the interference experiment to see the nonlocal exchange of modular momentum. *Nonlocal dynamical* quantum effects have no classical analogues.

Problems

12.1 Show that if the state $|\Psi(\mathbf{R})\rangle$ is normalized, then $\langle\Psi(\mathbf{R})|(\partial/\partial R_j)|\Psi(\mathbf{R})\rangle$ is imaginary, for each j.

12.2 Show that if we redefine the state $|\Psi_i(\mathbf{R})\rangle$ as

$$|\Psi_i(\mathbf{R})\rangle \to |\Psi_i'(\mathbf{R})\rangle = e^{i\Lambda(\mathbf{R})}|\Psi_i(\mathbf{R})\rangle \ ,$$

where $\Lambda(\mathbf{R})$ is any smooth scalar function of $\mathbf{R}$, the Berry connection transforms according to

$$\mathbf{A}_B \to \mathbf{A}_B' = \mathbf{A}_B - \nabla_{\mathbf{R}}\Lambda(\mathbf{R}) \ .$$

Show that the Berry phase is invariant under this redefinition of the state $|\Psi_i(\mathbf{R})\rangle$.

*12.3 For nondegenerate states $|\Psi_i(\mathbf{R})\rangle$, prove that the induced vector potential $\mathbf{A}$ satisfying Eqs. (12.5–6) yields the same induced field $\mathbf{B}$ as the Berry connection $\mathbf{A}_B \equiv \langle\Psi_i(\mathbf{R})|i\nabla_{\mathbf{R}}|\Psi_i(\mathbf{R})\rangle$, i.e. that $\mathbf{A}$ and $\mathbf{A}_B$ are equivalent up to a gauge transformation.

*12.4 (a) Calculate the Berry connection for a spin-$1/2$ particle polarized along an axis $\mathbf{R}$; show that (up to a gauge transformation) it equals

$$A_\theta = 0 \ , \quad A_\phi = (\cos\theta - 1)/2 \ ,$$

where $R_z = R\cos\theta$ and $R_x + iR_y = Re^{i\phi}\sin\theta$. Show that the Berry phase for a loop in $\mathbf{R}$ is $-1/2$ times the solid angle subtended by the loop at the origin $\mathbf{R} = \mathbf{0}$.
(b) Show that the Berry connection is equivalent to the vector potential of a Dirac monopole of strength $1/2$ at the origin $\mathbf{R} = 0$. Where is the Dirac string?

12.5 What induces the magnetic monopole of Prob. 12.4 and Eq. (12.11)? Consider a Hamiltonian

$$H_f = \frac{\mu B\hbar}{2}\left[\sigma_x\cos(\omega t) + \sigma_y\sin(\omega t)\right] \ ,$$

corresponding to Eq. (12.9) with $R_z = 0$.
(a) Compute $H' = U^\dagger H_f U$ where U is a single-valued unitary transformation:

$$U = e^{-i(\sigma_z+1)\omega t/2} .$$

(b) Derive the Schrödinger equation for H'. Show that the direction of the spin deviates from the plane $R_z = 0$ by an amount proportional to ω, for $\omega \ll \mu B$.
(c) The Born–Oppenheimer approximation takes $\mathbf{R}$ and $\mathbf{S} = \hbar\sigma/2$ to be parallel; yet they are not quite parallel and tend to align. Show [13] that the direction of this tendency, and its velocity dependence, match the force on a charge moving in the field of a monopole at the origin $\mathbf{R} = \mathbf{0}$.

*12.6 Prove that $F_{ij} \equiv -(i/\hbar)[P_i - A_i, P_j - A_j]$, where $\mathbf{P} - \mathbf{A}$ satisfies Eqs. (12.5–6), equals

$$F_{ij} = \frac{i}{\hbar} \sum_k \mathbf{\Pi}_k [A_i, A_j] \mathbf{\Pi}_k .$$

*12.7 Consider a Hamiltonian for a particle in a strong magnetic field:

$$H_f = \frac{\Omega}{2} \left[(p_x - \frac{y}{2})^2 + (p_y + \frac{x}{2})^2 \right] ,$$

where Ω is the Larmor frequency. The definitions $P \equiv p_x - y/2$, $Q \equiv p_y + x/2$, imply $[Q, P] = i\hbar$ and lead to a harmonic oscillator Hamiltonian:

$$H_f = \frac{\Omega}{2} \left(P^2 + Q^2 \right) .$$

The eigenstates are degenerate (Landau levels). To this Hamiltonian we add a perturbation, $H_s(x, y)$. If we consider $\mathbf{\Pi}_0$, the projector onto the ground state, Eqs. (12.5–6) become

$$[x - A_x, \mathbf{\Pi}_0] = 0 = [y - A_y, \mathbf{\Pi}_0] ,$$
$$\mathbf{\Pi}_0 A_x \mathbf{\Pi}_0 = 0 = \mathbf{\Pi}_0 A_y \mathbf{\Pi}_0 .$$

(a) Show that $A_x = Q$, $A_y = -P$ solves these equations.
(b) Show that if the perturbation is $H_s = \omega(x^2 + y^2)$, then the effective perturbation is a harmonic oscillator

$$\mathbf{\Pi}_0 H_s \mathbf{\Pi}_0 = [(x/2 - p_y)^2 + (y/2 + p_x)^2 + \hbar]\omega \mathbf{\Pi}_0 .$$

12.8 Prove that the limit

$$\lim_{N\to\infty} \left(1 + \frac{c_N}{N}\right)^N$$

approaches 1 if

$$\lim_{N\to\infty} c_N = 0 .$$

12.9 Consider a dense set of measurements of $\sigma_x \cos(\mu B t) - \sigma_y \sin(\mu B t)$ on a spin at times $t = n\pi/\mu BN$, where $n = 1, 2, 3, \ldots, N$. The initial state of the spin is $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ at time $t = 0$, and no field acts on the spin. (The Hamiltonian vanishes.) Show that, in the limit $N \to \infty$, each measurement of $\sigma_x \cos(\mu B t) - \sigma_y \sin(\mu B t)$ yields 1.

12.10 Suppose a normalized state $|\psi(t)\rangle$ evolves according to the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

such that $|\psi(T)\rangle = e^{i\varphi}|\psi(0)\rangle$, where φ is real. Define a set of states $|\tilde{\psi}(t)\rangle$

$$|\tilde{\psi}(t)\rangle = e^{-if(t)}|\psi(t)\rangle$$

such that $f(T) - f(0) = \varphi$.
(a) Show that

$$\varphi = \int_0^T dt \left[\langle \tilde{\psi}(t)| i \frac{d}{dt} |\tilde{\psi}(t)\rangle - \frac{1}{\hbar} \langle \tilde{\psi}(t)|H|\tilde{\psi}(t)\rangle \right] .$$

(b) Show that $\int_0^T dt \langle \tilde{\psi}|i(d/dt)|\tilde{\psi}\rangle$ depends only on the path of $|\tilde{\psi}(t)\rangle$ in Hilbert space, i.e. it is gauge invariant in the sense of Prob. 12.2.

12.11 Derive Eq. (12.19).

12.12 (a) From Eqs. (12.15–16), which define the phases Φ_B and Φ_D for an arbitrary path $\mathbf{R}[t]$, show that $\Phi_B + \Phi_D = \int_0^T L\{\mathbf{R}, d\mathbf{R}/dt\}dt$ where $L\{\mathbf{R}, d\mathbf{R}/dt\}$ contains time derivatives of first order only, i.e.

$$L\{\mathbf{R}, d\mathbf{R}/dt\} = f_0(\mathbf{R}) + \sum_i f_i(\mathbf{R}) dR_i/dt$$

for some functions f_0 and f_i ($i = 1, 2, 3, \ldots$). Obtain the equations of motion for $\mathbf{R}$ from $L\{\mathbf{R}, d\mathbf{R}/dt\}$.
(b) Let $H = \mu B \hbar \sigma_z/2$, where μB is a constant. Define $\mathbf{n}$ to be a unit vector and $|\mathbf{n}\rangle$ to be a normalized eigenstate of $\mathbf{n} \cdot \sigma$ with eigenvalue 1. Calculate Φ_D and Φ_B (Eqs. (12.15–16)) for an arbitrary path in θ and ϕ, where θ and ϕ are angular spherical coordinates for $\mathbf{n}$. Show that $\Phi_B + \Phi_D = \int_0^T L\{\theta, d\phi/dt\}dt$ where $L\{\theta, d\phi/dt\}$ equals

$$L\{\theta, d\phi/dt\} = -\frac{\mu B}{2} \cos\theta - \frac{1}{2}(1 - \cos\theta)\frac{d\phi}{dt}$$

up to a total time derivative.
(c) Apply Lagrange's equations to $L\{\theta, d\phi/dt\}$ and show that the spin precesses with constant θ at constant angular frequency μB.

12.13 (a) Prove Eq. (12.21).
(b) Let $n_1 = \sin\theta\cos\phi$, $n_2 = \sin\theta\sin\phi$ and $n_3 = \cos\theta$, and let $|\mathbf{n}\rangle$ be the normalized eigenstate of $\mathbf{n}\cdot\sigma$ with eigenvalue 1. The states $|\mathbf{n}\rangle$ form an overcomplete set. Prove that

$$\frac{1}{2\pi}\int d\phi d\theta \sin\theta |\mathbf{n}\rangle\langle\mathbf{n}| = 1 \ .$$

(c) Consider a particle with spin j, and define the state $|\phi\rangle$:

$$|\phi\rangle = \sum_{m=-j}^{j} e^{im\phi}|m\rangle \ ,$$

where $S_z|m\rangle = m\hbar|m\rangle$. Construct the identity operator from the projection operators $|\phi\rangle\langle\phi|$.

*12.14 (a) Derive the Feynman path integral, Eq. (12.23), from Eq. (12.22), and show that the normalization of Eq. (12.23) is $[mN/ihT]^{3(N-1)/2}$.
(b) Prove that typical paths in this path integral are those for which $[\mathbf{X}(t_{n+1}) - \mathbf{X}(t_n)]^2$ is of order T/N, for all n.

*12.15 A *classical* adiabatic theorem [14] applies to certain classical systems that depend on time through slowly changing parameters $\mathbf{R}$. The classical theorem applies to motion that is periodic in phase space. We can transform such motion into action I and angle ϕ variables that are canonically conjugate. In the adiabatic limit, I is a constant of the motion, the analogue of the energy quantum number. Hannay's angle is an extra shift in ϕ that a system acquires during a complete circuit in $\mathbf{R}$, relative to the time integral of the instantaneous angular frequency. The classical analogue of the Born–Oppenheimer approximation applies to coupled fast and slow motion, where the fast motion is periodic up to gradual changes due to the adiabatic change in $\mathbf{R}$. For a slow classical Hamiltonian $H_s = P^2/2M + V(\mathbf{R})$, the condition corresponding to Eq. (12.5) is [15]

$$\{\mathbf{P} - \mathbf{A}, I\} = 0 \ .$$

To fix $\mathbf{A}$ we need a condition analogous to Eq. (12.6). Eq. (12.6) requires that the expectation value of $\mathbf{A}$ vanish in all the subspaces. The classical analogue is that $\langle\mathbf{A}\rangle$, the average of $\mathbf{A}$, must vanish in averaging over the angle variable ϕ.
(a) Take $H_f = \mu B\mathbf{R}\cdot\mathbf{S}/R$ where the classical spin $\mathbf{S}$ obeys the Poisson relations

$$\{S_i, S_j\} = \sum_k \epsilon_{ijk}S_k$$

Show that the action is $I = \mathbf{R}\cdot\mathbf{S}/R$ and that $H_f = \mu BI$.
(b) Show that $\mathbf{A} = \mathbf{R}\times\mathbf{S}/R^2$ satisfies $\{\mathbf{P} - \mathbf{A}, I\} = 0$ and $\langle\mathbf{A}\rangle = 0$.
(c) Show that the effective adiabatic Hamiltonian $\langle H\rangle$ for this system is $\mu BI + (\mathbf{P} - \mathbf{A})^2/2M + (S^2 - I^2)/2MR^2 + V(\mathbf{R})$.
(d) Obtain the field corresponding to $\mathbf{A}$ by differentiating the Poisson brackets among components of $\mathbf{P} - \mathbf{A}$ with respect to I; show that it corresponds to a monopole located at $\mathbf{R} = \mathbf{0}$.

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13 Charges and Fluxons

Previous chapters of this book treat quantum measurements within nonrelativistic quantum mechanics, where they correspond well with what experiments can measure. In the next chapter, quantum measurements go relativistic – and the correspondence breaks down. Here, however, we take a break from quantum measurements. This chapter is exceptional in another respect, too. While other chapters begin with quantum paradoxes, this one begins with a classical (relativistic) paradox [1]. Although the paradox is entirely classical, its discussion and resolution came as late as 1967. No one seems to have noticed it before. The paradox is crucial for clarifying the entirely *quantum* interactions of "fluxons" and charges – the generalized Aharonov–Bohm effect of this chapter.

13.1 Hidden Momentum?

Consider two infinitely long, concentric cylinders. One fits inside the other, but the cylinders have nearly the same radius r_c. (See Fig. 13.1.) The cylinders carry equal and opposite charge, uniformly distributed, and have the same mass. They rotate in opposite senses, with equal

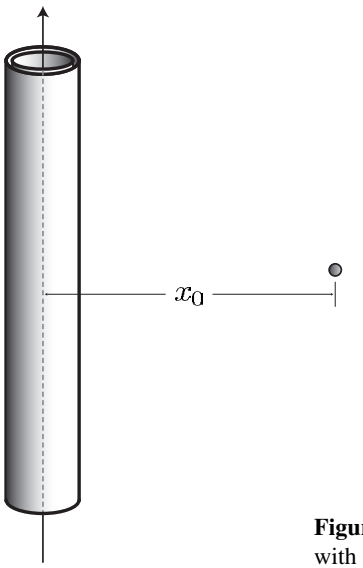

Figure 13.1: Two oppositely charged, concentric rotating cylinders, with a heavy particle a distance x_0 from their common axis of rotation.

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angular speeds; hence they have zero total angular momentum. At a distance $x_0 > r_c$ from the common axis of the cylinders, there is a heavy particle with charge Q. It does not move relative to this axis. Now suppose that the cylinders gradually stop rotating, because of slight friction between them. What happens?

Initially, the rotating cylinders enclose a uniform magnetic field $\mathbf{B}$. This magnetic field slowly drops to zero as the cylinders stop rotating. Maxwell's equations imply that the changing magnetic flux generates an electric field that circulates around the cylinders; this circulating electric field makes the charged particle accelerate. Now that is odd: the linear momentum of the particle was zero, and it is no longer zero; the total linear momentum of the cylinders was zero and it remains zero. Yet no external forces act on the charged particle and the cylinders. What happened to conservation of momentum?

Are we perhaps missing something? According to Poynting's theorem, there is momentum in the electromagnetic field, too. The field momentum $\mathbf{P}_{field}$ is

$$\mathbf{P}_{field} = \frac{1}{4\pi c}\int (\mathbf{E}\times\mathbf{B}) d^3x \ . \tag{13.1}$$

To calculate $\mathbf{P}_{field}$, let the z-axis be the common axis of the cylinders and let the charged particle lie initially on the x-axis. If the charge is far from the cylinders – i.e., if $x_0 \gg r_c$ – then inside the cylinders, the electric field $\mathbf{E}$ due to the charge is essentially independent of x and y (though dependent on z). The initial electromagnetic momentum points along the y-axis, as does the induced momentum of the charged particle. Do the electromagnetic and mechanical momenta have the same magnitude, too? The only component of the electric field contributing to $\mathbf{P}_{field}$ is E_x, and at a point on the z-axis it has magnitude

$$E_x = \frac{Qx_0}{(x_0^2+z^2)^{3/2}} \ ;$$

hence the magnitude of $\mathbf{P}_{field}$ is

$$P_{field} = \frac{\pi {r_c}^2 Q x_0 B}{4\pi c}\int_{-\infty}^{\infty}\frac{dz}{(x_0^2+z^2)^{3/2}} = \frac{QB{r_c}^2}{2x_0 c} \ . \tag{13.2}$$

On the other hand, Maxwell's equations yield the magnitude of the transient electric field at the position of the charge:

$$2\pi x_0 E_y = -\frac{1}{c}\frac{d\Phi}{dt} \ ,$$

where Φ is the magnetic flux in the cylinders. The force on the charged particle is QE_y, so the momentum of the charge, after the cylinders have stopped rotating, has magnitude

$$P_y = \int_{-\infty}^{\infty} dt Q E_y = \frac{QB{r_c}^2}{2x_0 c} \ , \tag{13.3}$$

and equals P_{field}. Thus we have confirmed the conservation of momentum. Odd or not, the charged particle *has* to accelerate in order to conserve the initial momentum in the electromagnetic field, as the cylinders stop rotating. Indeed, this example shows why there must be momentum in the electromagnetic field [2].

However, the closer we look at this resolution of the paradox, the less satisfactory it appears. Initially, the cylinders rotate and the charge is a distance x_0 from the axis of the cylinders, and stationary. We calculated that the momentum of the electromagnetic field is $QBr_c{}^2/2x_0c$ and points along the y-axis. Now let us move the charge along the x-axis to a new position a distance x_0' from the axis of the cylinders. We move it as slowly as we like, so that the charge scarcely induces a magnetic field; meanwhile, the magnetic flux in the cylinders does not change (if we neglect the friction between the cylinders). Then, when the charge has reached its new position, the momentum of the electromagnetic field has changed to $QBr_c{}^2/2x_0'c$ (along the y-axis). But the charge has not acquired any momentum along the y-axis. What happened to conservation of momentum?

We have assumed that the cylinders are infinitely long. Could the paradox be due to this unphysical assumption? Prob. 13.1 implies that it is not. Moreover, this paradox has another formulation, based on the following theorem: The total momentum in any stationary, localized distribution of matter is zero. For example, Fig. 13.2 shows a toroidal solenoid with a positively charged particle at its center; the charged particle does not move, and the current in the solenoid is constant, so this matter distribution is stationary and localized. There are no infinitely long cylinders here. But the momentum in the electromagnetic field need not vanish. If the electromagnetic momentum does not vanish, how can the total momentum vanish?

To prove the theorem, we define the stress (or energy-momentum) tensor $T^{\mu\nu}$, a function of space and time. The components of $T^{\mu\nu}$ represent local densities of energy and momentum and their fluxes: T^{00}/c^2 is the local density of mass, T^{0i}/c is the local density of the i-th component of momentum, and T^{ij} is the local flux density in the direction j of the i-th component of momentum. (Here $1 \le i, j \le 3$.) The total stress tensor includes the energy and momentum in the electromagnetic field as well as any mechanical energy and momentum. From local conservation of energy and momentum, we infer that $T^{\mu\nu}$ satisfies the equation

$$\sum_{\mu=0}^{3} \partial_\mu T^{\mu\nu} = 0 \,, \tag{13.4}$$

where $\partial_0 \equiv \partial_t/c \equiv (1/c)\partial/\partial t$ and $\partial_i \equiv \partial/\partial x_i$. Equation (13.4) is analogous to the continuity equation for electric charge and current. (See Prob. 13.2.)

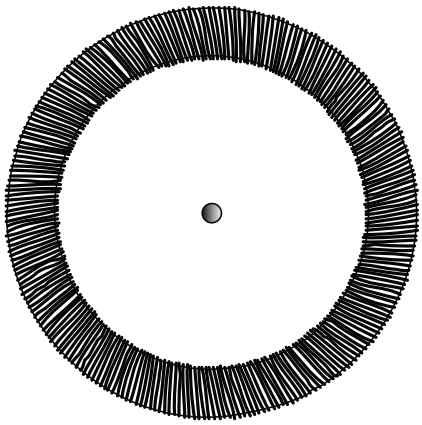

Figure 13.2: A toroidal solenoid with a positive charge at its center.

Since the matter distribution is localized, mechanical energy and momentum vanish beyond a certain distance. Also, electric and magnetic fields of localized charges and currents fall off according to the square of the distance, or faster, at large distances; hence $T^{\mu\nu}$ falls off according to the fourth power of the distance, or faster. We now compute the total mechanical and electromagnetic momentum of the matter distribution. It is the integral over all space of the momentum density:

$$\begin{aligned} P^i &= \frac{1}{c}\int T^{0i} d^3x \\ &= \sum_{j=1}^{3}\left[\int \partial_j \left(T^{0j}x^i\right) d^3x - \int x^i \partial_j T^{0j} d^3x\right] . \end{aligned} \tag{13.5}$$

The second line of Eq. (13.5) contains two integrals. The first integral reduces (by the divergence theorem) to a surface integral:

$$\sum_{j=1}^{3}\int \partial_j \left(T^{0j}x^i\right) d^3x = \sum_{j=1}^{3}\int T^{0j}x^i dS^j ,$$

where dS^j is an oriented area element for the surface bounding the volume of integration. This integral must vanish because, at large distances, the area of the surface grows as the square of the distance, while the integrand falls off as the third power of the distance or faster. The second integral vanishes because the matter distribution is stationary, i.e. $\partial_0 T^{\mu\nu} = 0$; then Eq. (13.4) implies that $\sum_j \partial_j T^{0j} = 0$. Hence from Eq. (13.5) we infer that the total momentum P^i is zero.

Now back to Fig. 13.2. Let us suppose that the coils of the solenoid are not conducting wires; rather, they are neutral glass tubes through which a gas of electrons flows. The tubes are covered on the outside with a positive surface charge that cancels the electric field of the electrons. A dilute gas of electrons does not screen the electric field of the positive charge at the center of the solenoid. Hence the electromagnetic momentum of this configuration is not zero; the crossed electric and magnetic fields within the solenoid yield an electromagnetic momentum parallel to the axis of symmetry of the solenoid. According to our theorem, then, the total mechanical momentum cannot be zero. But where is this mechanical momentum?

13.2 Duality of the Aharonov–Bohm Effect

Let's try out a model Lagrangian for an electron and an infinitely thin solenoid or idealized flux line – a "fluxon". For simplicity, let space be two-dimensional; then we can represent both the fluxon and the electron as points on a plane, with the magnetic field vanishing everywhere except at the location of the fluxon. (We have not changed the topology – a closed electron path has the same winding number whether it winds around a line in three dimensions or a point in two dimensions.) Let $\mathbf{R}$ represent the position of the fluxon, and $\mathbf{r}$ the position of the electron. If position of the fluxon is fixed, a Lagrangian for the electron is

$$L = \frac{m}{2}\dot{r}^2 + \frac{e}{c}\mathbf{A}(\mathbf{r}-\mathbf{R})\cdot\dot{\mathbf{r}} , \tag{13.6}$$

where $\dot{\mathbf{r}} = d\mathbf{r}/dt$ and $\mathbf{A}(\mathbf{r}-\mathbf{R})$ is a vector potential, a function of the relative distance $\mathbf{r}-\mathbf{R}$. This Lagrangian leads to the correct equation of motion for the electron,

$$m\ddot{r}_j = m\frac{d}{dt}\dot{r}_j = \frac{e}{c}\sum_i \dot{r}_i(\partial_j A_i - \partial_i A_j) \ ,$$

with $m\ddot{\mathbf{r}}$ equal to the Lorentz force. What if the fluxon moves, too? We can modify Eq. (13.6) by adding a kinetic term for the fluxon,

$$L = \frac{m}{2}\dot{r}^2 + \frac{M}{2}\dot{R}^2 + \frac{e}{c}\mathbf{A}(\mathbf{r}-\mathbf{R})\cdot\dot{\mathbf{r}} \ , \tag{13.7}$$

but we obtain an incorrect equation of motion for the electron:

$$m\ddot{r}_j = \frac{e}{c}\sum_i \dot{r}_i(\partial_j A_i - \partial_i A_j) + \frac{e}{c}\sum_i \dot{R}_i\partial_i A_j \ .$$

It contains, in addition to the Lorentz force, a force that is not gauge invariant, acting on the electron even where the magnetic field vanishes. We can see what is wrong directly from the Lagrangian. Suppose that for $\mathbf{A}$ in Eq. (13.7) we substitute $\mathbf{A} = \nabla_{\mathbf{r}}\Lambda$ for some single-valued scalar function $\Lambda(\mathbf{r}-\mathbf{R})$, so the magnetic field vanishes everywhere (even at the location of the fluxon); then there is no interaction, and the interaction term in Eq. (13.7) should reduce to a total time derivative. But $\nabla_{\mathbf{r}}\Lambda(\mathbf{r}-\mathbf{R})\dot{\mathbf{r}}$ is not a total time derivative. On the other hand, $\nabla_{\mathbf{r}}\Lambda(\mathbf{r}-\mathbf{R})(\dot{\mathbf{r}}-\dot{\mathbf{R}})$ *is* a total time derivative. If we replace $\dot{\mathbf{r}}$ by $\dot{\mathbf{r}}-\dot{\mathbf{R}}$ in this term we obtain

$$L = \frac{m}{2}\dot{r}^2 + \frac{M}{2}\dot{R}^2 + \frac{e}{c}\mathbf{A}(\mathbf{r}-\mathbf{R})\cdot(\dot{\mathbf{r}}-\dot{\mathbf{R}}) \ . \tag{13.8}$$

Now Eq. (13.8) implies a gauge-invariant force that vanishes along with the magnetic field:

$$m\ddot{\mathbf{r}} = \frac{e}{c}(\dot{\mathbf{r}}-\dot{\mathbf{R}})\times\mathbf{B} = -M\ddot{\mathbf{R}} \ , \tag{13.9}$$

where $\mathbf{B} = \nabla_{\mathbf{r}}\times\mathbf{A}(\mathbf{r}-\mathbf{R})$. If Eq. (13.9) is valid, the electron and the fluxon exert a force on each other only when they touch, and their total momentum is a constant of the motion.

Is Eq. (13.9) valid? As an equation of motion for a classical model, it is certainly valid. A subtler question is whether Eqs. (13.8–9) yield a valid *quantum* model. Equations (13.8–9) treat the electron and fluxon on an equal footing. We know that an electron diffracting around a fluxon acquires a topological phase. This model therefore implies that a fluxon diffracting around an electron acquires a topological phase. Does it?

Consider an electron passing a stationary fluxon. If the electron is a quantum particle, it can divide into two wave packets as it passes the fluxon. Let the fluxon, too, divide into two wave packets, before the electron passes, such that only one of the two fluxon wave packets lies between the two electron wave packets as they pass. (See Fig. 13.3.) Now let two experimenters, Alice and Bob, observe the electron and the fluxon, respectively. After the electron passes, they each perform one of two measurements. Alice can perform a measurement that reveals which path the electron took, or a measurement that reveals the relative phase between the electron wave packets. These complementary measurements are analogous to measurements

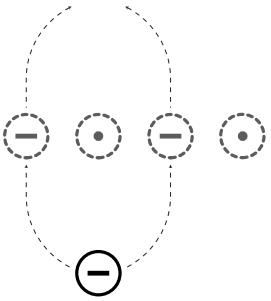

Figure 13.3: An electron and a fluxon, each in a superposition of two wave packets; the electron wave packets encircle only one of the fluxon wave packets.

in a double-slit experiment. Similarly, Bob can measure either the position of the fluxon or the relative phase of the fluxon wave packets, after the electron goes by. Let us anticipate the results of their measurements.

At the beginning of the experiment, the state $|\Psi_{in}\rangle$ of the fluxon and the electron is a product state:

$$|\Psi_{in}\rangle = \frac{1}{2}(|f_1\rangle + |f_2\rangle) \otimes (|e_1\rangle + |e_2\rangle) \ ,$$

where $|f_1\rangle$ and $|f_2\rangle$ represent the two fluxon wave packets and $|e_1\rangle$ and $|e_2\rangle$ represent the two electron wave packets. The state $|\Psi_{fin}\rangle$, after the electron has passed the fluxon, is no longer a product state; the relative phase between $|e_1\rangle$ and $|e_2\rangle$ depends on the fluxon position:

$$|\Psi_{fin}\rangle = \frac{1}{2}|f_1\rangle \otimes (|e_1\rangle + |e_2\rangle) + \frac{1}{2}|f_2\rangle \otimes (|e_1\rangle + e^{i\phi_{AB}}|e_2\rangle) \ .$$

Here ϕ_{AB} is the Aharonov–Bohm phase, and $|f_2\rangle$ represents the fluxon positioned between the two electron wave packets. Now suppose that Bob measures the position of the fluxon and Alice measures the relative phase of the electron. By repeating this experiment many times and comparing their results, Alice and Bob confirm the Aharonov–Bohm effect: Alice finds the relative phase ϕ_{AB} if and only if Bob finds the fluxon between the two electron paths.

However, we can rewrite $|\Psi_{fin}\rangle$ as follows:

$$|\Psi_{fin}\rangle = \frac{1}{2}(|f_1\rangle + |f_2\rangle) \otimes |e_1\rangle + \frac{1}{2}(|f_1\rangle + e^{i\phi_{AB}}|f_2\rangle) \otimes |e_2\rangle \ .$$

This simple rewriting has a profound consequence. Now suppose that Bob measures the relative phase of the fluxon and Alice measures the position of the electron. They will discover an effect that is dual to the Aharonov–Bohm effect: the *fluxon* acquires the phase ϕ_{AB} if and only if the *electron* passes between the two fluxon wave packets. To complete the duality, we can choose a reference frame in which the fluxon passes by the stationary electron. (The relative phase between two wave packets cannot depend on the reference frame.) Then we find the same relative phase whether the electron paths enclose the fluxon or the fluxon paths enclose the electron.

This duality is just what Eqs. (13.8–9) imply. If an electron acquires a phase by encircling a stationary fluxon, a fluxon acquires the same phase by encircling a stationary electron. Equations (13.8–9) are just a two-dimensional model, but Eq. (13.9) is gauge invariant, and it

implies (correctly) that the electron and the fluxon, when they do not touch, share a phase. These features make Eqs. (13.8–9) a model worth studying, a model combining the Aharonov–Bohm effect with Berry's phase.

13.3 The Aharonov–Bohm Effect and Berry's Phase

An electron winding around a solenoid acquires an Aharonov–Bohm phase. This phase is topological: it is proportional to the winding number of the electron path around the solenoid. If the enclosed flux is Φ and the path winds n times around the solenoid, the Aharonov–Bohm phase is $\phi_{AB} = ne\Phi/\hbar c$. But what if the solenoid is neither inside the closed electron path nor outside it, but on it? The question makes no sense if the electron is a classical point charge, because – assuming that the electron does not penetrate the solenoid – it must pass the solenoid on one side or the other. But the Aharonov–Bohm effect is a quantum effect, and a *quantum* electron propagates as a wave packet. This electron "cloud" could pass on both sides of a well shielded solenoid without ever penetrating it. Suppose it does; what would the Aharonov–Bohm phase be?

We can apply the model of the last section, Eq. (13.8), to this question. Treat the solenoid as a fluxon that moves while remaining perpendicular to a fixed plane, and an electron that moves in the plane. The effective Hamiltonian in two dimensions is

$$H = \frac{(\mathbf{p} - e\mathbf{A}/c)^2}{2m} + V(\mathbf{r}) + \frac{(\mathbf{P} + e\mathbf{A}/c)^2}{2M} \ , \tag{13.10}$$

where $\mathbf{p}$ and m are the momentum and mass of the electron, respectively, and $\mathbf{P}$ and M are the momentum and mass of the fluxon. For the vector potential $\mathbf{A}$ we take

$$\mathbf{A} = \frac{\Phi}{2\pi} \nabla_{\mathbf{r}} \theta(\mathbf{r} - \mathbf{R}) \ , \tag{13.11}$$

where $\theta(\mathbf{r} - \mathbf{R})$ is the (multivalued) angle of the vector $\mathbf{r} - \mathbf{R}$ relative to a fixed vector in the plane, and Φ is the flux of the fluxon. Equation (13.10) includes a binding potential $V(\mathbf{r})$ for the electron; it is convenient to bind the electron in a fixed potential. What happens if the fluxon passes through the electron cloud? We cannot give a general answer, but if the fluxon moves slowly, we can study this model in the adiabatic approximation. (See Sect. 12.2.) If, in addition, the fluxon carries flux $\Phi = hc/2e$ – i.e., if it is a *semifluxon* – the analysis of the model becomes simple and elegant.

Why a semifluxon? To see what makes a semifluxon special, let us define a rather artificial time-reversal operation T sending $\mathbf{p} \to -\mathbf{p}$, $\mathbf{P} \to -\mathbf{P}$. (It is artificial because a true time-reversal operation would send $\mathbf{A} \to -\mathbf{A}$, as well. See Sect. 10.2.) In general, T is not a symmetry of Eq. (13.10), because $\mathbf{A} \neq -\mathbf{A}$. But in the special case of a semifluxon, the difference between $\mathbf{A}$ and $-\mathbf{A}$ is $\mathbf{A} - (-\mathbf{A}) = 2\mathbf{A} = (\hbar c/e)\nabla_{\mathbf{r}}\theta(\mathbf{r} - \mathbf{R})$; it has the same form as Eq. (13.11) but corresponds to $\Phi = hc/e$ – one flux quantum – at $\mathbf{r} = \mathbf{R}$. Any multiple of the flux quantum is physically indistinguishable from no flux, hence $\mathbf{A} \to -\mathbf{A}$ is a pure gauge transformation. (See also Prob. 13.5.)

It follows that T is a symmetry of H, in some gauge.[1] Now let the semifluxon slowly move along a closed loop C. The adiabatic motion alters the state of the electron by an overall phase factor, which in general includes a Berry phase $\Phi_B(C)$ as well as a dynamical phase Φ_D. Time-reversal symmetry implies that the state acquires the same overall phase factor $e^{i\Phi_D}e^{i\Phi_B(C)}$ if the semifluxon moves along C in the opposite direction. (See Eq. (10.4).) The dynamical phase is the same in the two cases, but the Berry phase changes sign. (See Eq. 12.3.) Since the overall phase factor is the same, we have

$$e^{i\Phi_D}e^{i\Phi_B(C)} = e^{i\Phi_D}e^{-i\Phi_B(C)} ;$$

hence the Berry phase $\Phi_B(C)$ can only be 0 or π (up to a multiple of 2π).

Now that is a surprise. What becomes of the phase $\Phi_B(C)$ as we deform the path C? Suppose the electron cloud lies entirely within a finite region $\mathcal{S}$. Figure 13.4 shows a closed fluxon path C_1 outside the region $\mathcal{S}$ without enclosing it. For this path, both the Aharonov–Bohm phase and Berry's phase vanish. Now let us deform the path C_1 into a large loop C_2 that encloses the region $\mathcal{S}$ without touching it. For this path Berry's phase equals π and it is entirely the Aharonov–Bohm phase, since the path encloses all the charge.[2] We can distort C_1 into C_2 in many steps, enlarging the loop by an infinitesimal region at each step. We might have expected the phase Φ_B of the loop to rise gradually from 0 to π – but we have discovered that Φ_B can only be 0 or π. It follows that some infinitesimal region contains a "singular point" $\mathcal{P}$ such that Φ_B jumps when we annex this infinitesimal region. But only degeneracy between states of the electron can cause such a jump in Berry's phase. (See Prob. 13.8.) Hence for any binding potential $V(\mathbf{r})$ there exists a point $\mathcal{P}$ such that, if a semifluxon is introduced there, the state of the electron becomes degenerate with another state.

We see that, in the adiabatic limit, the Aharonov–Bohm phase and Berry's phase combine is a subtle way. We referred to $\Phi_B(C)$ as Berry's phase; but when the loop C encloses the entire electron cloud without touching it, $\Phi_B(C)$ is simply the Aharonov–Bohm phase. When C is an infinitesimal loop about a point of degeneracy, $\Phi_B(C)$ appears as Berry's adiabatic phase; but even for an infinitesimal loop, we can interpret $\Phi_B(C)$ as the Aharonov–Bohm phase. (See Prob. 13.6.) What is striking is that the Aharonov–Bohm and Berry phases combine in a *topological* phase that depends only on the winding number of the fluxon path around the point $\mathcal{P}$ in the electron cloud.

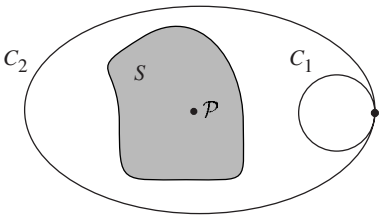

Figure 13.4: An electron cloud with support in a region $\mathcal{S}$ and two possible paths, C_1 and C_2, of a semifluxon. At the point $\mathcal{P}$, the semifluxon induces a degeneracy in the energy of the electron.

[1] Explicitly, the gauge transformation that makes T a symmetry of H is $\mathbf{A} \to \mathbf{A} - \nabla_{\mathbf{r}}(\hbar c/2e)\bar{\theta}(\mathbf{r}-\mathbf{R})$, where $\bar{\theta}(\mathbf{r}-\mathbf{R})$ is the *single*-valued form of $\theta(\mathbf{r}-\mathbf{R})$ and jumps by 2π along a half-line of constant $(\mathbf{r}-\mathbf{R})/|\mathbf{r}-\mathbf{R}|$. (See Prob. 13.6.) In this gauge $\mathbf{A}$ vanishes except along the half-line.

[2] The Aharonov–Bohm phase, unlike Berry's phase, is defined even when the motion is not adiabatic; here, however, the motion is adiabatic and Berry's phase includes the Aharonov–Bohm phase.

13.4 The Aharonov–Casher Effect

The topology of the Aharonov–Bohm effect is the same in two dimensions as in three: a closed electron path has the same winding number whether it winds around a point in two dimensions or a line in three dimensions. So a problem involving electrons and fluxons in three dimensions often reduces to an effective problem in two dimensions. However, although there is just one way to go from three dimensions to two, there are two inequivalent ways to go back. The familiar way is to turn fluxons back into lines; but we could also turn electrons into lines of charge, leaving the fluxons as points. Both ways preserve the topology. What happens if we go the unfamiliar way?

In two dimensions, an electron and a (neutral) fluxon are dual. (See Sect. 13.2.) A closed fluxon path around the electron, and a closed electron path around the fluxon, yield the same phase ϕ_{AB}. If we go the unfamiliar way back to three dimensions, we arrive at a new topological effect, dual to the Aharonov–Bohm effect in three dimensions: a neutral "fluxon particle", diffracting around a charged wire, should acquire a topological phase, just like a charged particle diffracting around a line of flux [3]. The neutron, with its intrinsic magnetic moment, is an example of a fluxon particle. (But unlike a solenoid, a neutron has a nonvanishing field around it.) Hence a neutron should acquire a topological phase when it diffracts around a line of charge. This effect, the Aharonov–Casher effect, has been observed experimentally [4].

Topology aside, are the two effects truly dual? What is so striking about the Aharonov–Bohm effect is that, even if the magnetic field of the solenoid vanishes all along the path of the electron, the electron acquires an observable phase. In the Aharonov–Casher effect, the neutron plows right through the electric field of the line of charge. The two effects are not similar in this respect. The question, however, is whether the electric field acts on the neutron. Does the line of charge exert a force on the neutron? If – and only if – it exerts no force on the neutron are the Aharonov–Bohm and Aharonov–Casher effects truly dual.

To address this question, consider an infinitely long, straight solenoid that passes an infinitely long, straight line of charge; the solenoid and the line of charge remain parallel at all times. (See Fig. 13.5.) We know that the solenoid and the line of charge exert no force on each other. (See Sect. 13.2.) Now, a solenoid is equivalent to an infinite stack of identical current

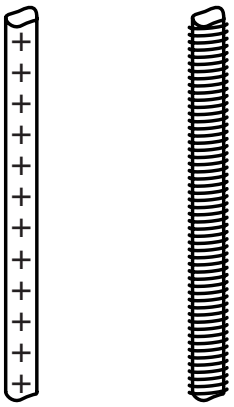

Figure 13.5: A line of charge and a solenoid, both straight and infinitely long, parallel to each other.

loops; each current loop is a magnetic dipole, like a neutron. Suppose the line of charge exerts a force on one of the current loops in the stack. By symmetry, it exerts the *same* force on *all* the current loops in the stack. But the sum of these forces vanishes: the line of charge exerts no force on the solenoid itself. We conclude, therefore, that an infinitely long, straight line of charge exerts no force on a current loop with polarization parallel to the line of charge.[3]

This conclusion seems, at first, paradoxical. The line of charge is the source for an electric field $\mathbf{E}$, while the magnetic field $\mathbf{B}$ vanishes. But since the neutron moves relative to the line of charge, the neutron encounters a magnetic field $\mathbf{B}'$; $\mathbf{B}'$ is the Lorentz transform of $\mathbf{E}$ to the rest frame of the neutron. The Lorentz transformation for $\mathbf{B}'$ is [5]

$$\mathbf{B}' = \gamma(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}) - \frac{\gamma^2}{c^2(\gamma+1)}\mathbf{v}(\mathbf{v}\cdot\mathbf{B}) = -\gamma\frac{\mathbf{v}}{c} \times \mathbf{E} \ ,$$

where $\mathbf{v}$ is the velocity of the neutron with respect to the line of charge and $\gamma = (1-v^2/c^2)^{-1/2}$. For the force $\mathbf{F}$ on a current loop with magnetic dipole moment μ in a magnetic field $\mathbf{B}$ we have a familiar formula:

$$\mathbf{F} = \nabla(\mu \cdot \mathbf{B}) \ , \tag{13.12}$$

so the force on the current loop does *not* vanish – $\mu \cdot \mathbf{B}'$ is not constant in space.

We conclude that there is no force on a moving current loop – and also that there is a force! Which conclusion is right? In trying to resolve this paradox, we might recall the paradox of Sect. 13.1. Both versions of the paradox of Sect. 13.1 imply that there can be mechanical momentum that is hidden from us – so far, at least. If hidden momentum resolves the paradox of Sect. 13.1, perhaps hidden momentum resolves the paradox here, as well. So let us return to the toroidal solenoid with the positive charge at its center, in Fig. 13.2. Where could the mechanical momentum be hidden? For mechanical momentum, there must be moving parts. The positive charge does not move, but the solenoid has moving parts: there are electrons flowing in the tubes of the solenoid. The electrons have momentum, and they respond to the positive charge by speeding up on their way in (towards the axis of symmetry of the solenoid) and slowing down on their way out. Hence, their momentum on the inner side of the torus exceeds their momentum on the outer side of the torus, and there is net mechanical momentum along the axis of the torus.

Eureka! This must be the resolution of the paradox. The only problem with this resolution is that a simple calculation disproves it. Consider a single tube in the solenoid. The linear momentum density of the electrons in the tube is $m\mathbf{v}\rho/e$, where m is the electron mass, ρ is the linear charge density and $\mathbf{v}$ is the velocity of the electrons. But ρv equals the current I in the tube. The current I must be the same all along the tube. (If it were not, charge would build up somewhere in the tube; but the charge density everywhere is constant in time, by assumption.) Since ρv is the same all along the tube, so is the magnitude of the momentum density. Hence the total mechanical momentum of the particles vanishes.

As Einstein remarked, God is subtle but not malicious. Our simple calculation was too simple: we used the nonrelativistic formula for mechanical momentum, yet the momentum

[3] This conclusion holds only if the line of charge is straight. By contrast, even a kinky solenoid exerts no force on a passing electron.

in the electromagnetic field vanishes in the limit $c \to \infty$. Let us try the relativistic formula. The current $I = \rho v$ is still the same all along the tube, but now the momentum density is not $m\mathbf{v}\rho/e$. The momentum $\mathbf{p}$ of each particle is related to its energy E according to the equation

$$\mathbf{p} = \frac{E}{c^2}\mathbf{v} \ .$$

Hence the momentum density of the particles is equal to $E\mathbf{v}\rho/c^2 e$; and E, unlike ρv, is not constant all along the tube. The total mechanical momentum of the particles does not vanish!

As a test of this resolution of the paradox, we roughly calculate the mechanical momentum and the momentum in the electromagnetic field. (For an exact calculation, see Prob. 13.9.) Suppose that the electric and magnetic field strengths in the solenoid are constant and that the solenoid cross section is a square of area L^2. Let N denote the number of current loops in the solenoid, and V the volume of the solenoid. The magnitude of the electromagnetic field momentum is

$$P_{field} = |\mathbf{E}||\mathbf{B}|V/4\pi c \ , \tag{13.13}$$

where the magnetic field strength $|\mathbf{B}|$ equals

$$|\mathbf{B}| = 4\pi INL^2/Vc \ . \tag{13.14}$$

Combining Eqs (13.13–14), we obtain $IL^2|\mathbf{E}|N/c^2$ for the momentum in the electromagnetic field. On the other hand, each electron in the solenoid has $e|\mathbf{E}|L$ more energy when it moves along the inner side (inner radius) of the solenoid than when it moves along the outer side (outer radius). Hence from the outer side of the solenoid to the inner side, its momentum increases by $ev|\mathbf{E}|L/c^2$. The density of particles in each tube is ρ/e, and there are N tubes in the solenoid, so the magnitude of the mechanical momentum is $IL^2|\mathbf{E}|N/c^2$, which coincides with the magnitude of the electromagnetic momentum. The momenta have opposite signs, however, and sum to zero.

So there is hidden mechanical momentum in the charged gas. Likewise, there is hidden mechanical momentum in the rotating charged cylinders of Sect. 13.1. (See Prob. 13.11.) For a current loop in a conductor the resolution differs qualitatively (since a conductor screens the electric field of an external charge) but not quantitatively [**?**]. As regards a magnetic dipole we can summarize the resolution as follows: the force on a magnetic dipole μ at rest in fields $\mathbf{E}$ and $\mathbf{B}$ is

$$\mathbf{F} = \nabla(\mu \cdot \mathbf{B}) - \frac{1}{c}\frac{d}{dt}(\mu \times \mathbf{E}) \ . \tag{13.15}$$

For any model of a magnetic dipole involving current loops (but no magnetic monopoles), Eq. (13.15) – and not Eq. (13.12) – is correct, in general. Equation (13.15) assumes that $\mathbf{E}$ and $\mathbf{B}$ are constant over the magnetic dipole, i.e. that the magnetic dipole is small. In Eq. (13.15), the term $-\mu \times \mathbf{E}/c$ represents the electromagnetic field momentum arising from the magnetic dipole and the field $\mathbf{E}$. (See Prob. 13.13.) For a magnetic dipole moving past a line of charge, with μ parallel to the line of charge, the two terms in Eq. (13.15) cancel, so the force on the dipole vanishes. (See Prob. 13.14.) Hence the Aharonov–Casher effect is indeed dual to the Aharonov–Bohm effect [7].

Problems

13.1 A particle of charge Q sits a distance x_0 from the axis of a straight flux line of negligible thickness and length l. Compute the momentum of the electromagnetic field, in the absence of screening, and show that it equals the momentum the particle acquires if the flux drops to zero. (See Sect. 13.1.) What is the magnetic field due to the ends of the solenoid?

13.2 Prove Eq. (13.4) for the $\nu = 1$ component of $T^{\mu\nu}$ by applying local conservation of momentum to the flux of momentum through an infinitesimal volume, and the divergence theorem.

13.3 (a) Derive Eq. (13.10) (without the potential $V(\mathbf{r})$) from Eq. (13.8).
(b) Show that the vector potential Eq. (13.11) corresponds to a singular flux Φ at the position $\mathbf{R}$ of the fluxon.

13.4 In two space dimensions, we can imagine *anyons*, particles carrying *any* charge and flux. What is the topological phase of an anyon of charge e_1 and flux f_1 winding n times around a stationary anyon of charge e_2 and flux f_2?

*13.5 "Objection! Section 13.3 states that $\mathbf{A} \to \mathbf{A} + (\hbar c/e)\nabla_{\mathbf{r}}\theta(\mathbf{r}-\mathbf{R})$ is a pure gauge transformation. But if the electron wave function $\Psi(\mathbf{r}-\mathbf{R})$ does not vanish at $\mathbf{r} = \mathbf{R}$, its gauge transform, $e^{i\theta(\mathbf{r}-\mathbf{R})}\Psi(\mathbf{r}-\mathbf{R})$, is not well defined!" Answer this objection as follows:
(a) Consider the Schrödinger equation $E\Psi(\mathbf{r}) = -(\hbar^2/2m)\nabla^2\Psi + V(r)\Psi$ in two dimensions, where $V(r)$ is infinite for $r \geq r_0$ and vanishes for $r < r_0$. Show that the solutions $\Psi(\mathbf{r})$ that are finite and nonzero at $\mathbf{r} = \mathbf{0}$ are $\Psi(\mathbf{r}) = NJ_0(kr)$ where $k = \sqrt{2mE}/\hbar$, J_0 is the Bessel function of order 0, N is a normalization constant and E satisfies $J_0(kr) = 0$.
(b) Now impose the boundary condition $\Psi(\mathbf{r}) = 0$ at $r = a$, to make room for a thin solenoid of radius a at the origin. Obtain a solution of the form

$$c_J J_0(k'r) + c_N N_0(k'r) \ ,$$

where $k' = \sqrt{2mE'}/\hbar$ and N_0 is the Neumann function (Bessel function of the second kind) of order 0.
(c) Use the small-ρ expansions

$$J_0(\rho) = 1 - \rho^2/4 + \dots \ , \quad N_0(\rho) = \frac{2}{\pi}\left(\ln\frac{\rho}{2} + C_{Euler}\right) + \dots$$

(where $C_{Euler} \approx 0.5772$), to show that c_N approaches $-\pi c_J/2\ln a$ as a approaches 0, that E' approaches E, and that $\Psi(\mathbf{r})$ approaches $NJ_0(kr)$ everywhere except for $r \approx a$. Explain how this solution overcomes the objection.

*13.6 Consider a Hamiltonian H_e for a bound electron in the presence of a semifluxon at the point $\mathbf{R}$:

$$H_e = (\mathbf{p} + e\mathbf{A}/c)^2 + V(\mathbf{r}) \ ,$$

where $\mathbf{A}$ is the vector potential in Eq. (13.11) with $\Phi = hc/2e$.
(a) Prove that a nondegenerate eigenfunction of H_e has a *null line*, i.e. a line on which the eigenfunction vanishes, and that this line stretches from the semifluxon to infinity.
(b) Show that if the semifluxon slowly crosses through the electron wave function, all the charge of the electron passes the semifluxon on one side only.

13.7 Apply a suitable gauge transformation to the Hamiltonian of Eqs. (13.10–11) and show that the eigenstates of the transformed Hamiltonian are eigenstates of the time-reversal operation T of Sect. 13.3.

13.8 Consider a Hamiltonian $H(x, y)$ depending on parameters x, y. The eigenstates of H are nondegenerate and discrete, but two of them become nearly degenerate in a small region. In this region, we truncate the Hilbert space for H to these two states and replace H with an effective Hamiltonian

$$H_{eff} = H_0(x, y) + H_1(x, y)\sigma_1 + H_2(x, y)\sigma_2 + H_3(x, y)\sigma_3$$

using the Pauli matrices σ_i.
(a) Find the condition for a degeneracy at the point (x_*, y_*) and show that, in general, there are no solutions. Show that if H_{eff} can be chosen real then there are, in general, isolated points of degeneracy.
(b) Consider a cyclic adiabatic evolution of H_{eff} in which the parameters x, y describe an *infinitesimal* loop about a point (x_0, y_0). Show that an eigenstate of H_{eff} acquires a nonzero Berry phase only if $(x_0, y_0) = (x_*, y_*)$.

13.9 Consider a stationary, localized current loop with current density $\mathbf{J}$ sitting in a static electric potential V. Show that the mechanical momentum,

$$\mathbf{P}_{mech} = \frac{1}{c^2} \int d^3x V \mathbf{J} \ ,$$

of the current loop is equal in magnitude to the electromagnetic momentum, Eq. (13.1), and opposite in sign. (Hint: use a vector identity and Maxwell's equations to prove

$$\nabla \times (V\mathbf{B}) = (4\pi/c)V\mathbf{J} - \mathbf{E} \times \mathbf{B} \ ,$$

where ϕ is the electrostatic potential and $\mathbf{J}$ is the current density.)

13.10 A particle carrying charge q winds around a solenoid and acquires an Aharonov–Bohm phase. The solenoid is made of conducting coils and screens the electric field of the particle; it thus acquires a dipole moment, and there is an electrostatic attraction between the solenoid and the particle. Find a limit in which the Aharonov–Bohm phase remains constant but the electrostatic attraction approaches zero.

*13.11 A particle of charge Q sits a distance x_0 from two long, oppositely charged rigid cylinders of nearly equal radius r_c that rotate with constant but opposite angular speed about their common axis, producing a magnetic field $\mathbf{B}$ inside. (See Sect. 13.1.)

(a) Calculate the hidden momentum of the cylinders and show that it equals $Q\mathbf{B}{r_c}^2/2x_0c$.
(b) The hidden momentum of the cylinders cancels the momentum of the electromagnetic field, Eq. (13.2), in keeping with the theorem in Sect. 13.1. But once the cylinders stop rotating, there is no momentum in the electromagnetic field and no hidden momentum. Yet there is the momentum P_y of the charged particle, Eq. (13.3). What happened to conservation of momentum? Explain.

*13.12 Next to a long conducting tube lies a heavy, uniform line of charge, parallel to the axis of the tube. A uniform current circulates in the tube and generates a magnetic flux through the tube. Since the tube screens the field of the line of charge, the momentum in the electromagnetic field vanishes. However (if the tube is not a superconductor) the current in the tube quickly decays, inducing a circulating electric field that acts on the line of charge. The line of charge acquires momentum. What happened to conservation of momentum? Compute the screening charge on a length L of the tube and show that it acquires a total momentum equal and opposite to the final momentum in a length L of the line of charge. Neglect end effects.

13.13 Consider a localized current distribution $\mathbf{J}$ in a uniform electric field $\mathbf{E}$. Prove that the momentum in the electromagnetic field is equal to

$$\mathbf{P}_{field} = \frac{1}{2c^2}\mathbf{E} \times \int (\mathbf{r} \times \mathbf{J}) d^3x \equiv \frac{1}{c}\mathbf{E} \times \mu \ ,$$

where μ is the magnetic moment of the current distribution.

13.14 Consider a magnetic dipole of dipole moment μ passing a straight line of charge, with μ parallel to the line of charge. Compute the electric field $\mathbf{E}'$ and magnetic field $\mathbf{B}'$ in the rest frame of the dipole and substitute them for $\mathbf{E}$ and $\mathbf{B}$, respectively, in Eq. (13.15). Show that the force $\mathbf{F}$ in Eq. (13.15) vanishes.

*13.15 A current loop with magnetic dipole moment μ, moving with velocity $\mathbf{v}$, passes a line of charge. The electric field $\mathbf{E}$ of the line of charge yields a magnetic field $\mathbf{B}' = -\mathbf{v} \times \mathbf{E}/(c^2 - v^2)^{1/2}$ in the rest frame of the current loop. Equation (13.12) (but not Eq. (13.15)) implies that the current loop feels a force $\nabla(\mu \cdot \mathbf{B}')$ in its rest frame. Such a force, if it existed, could account for the Aharonov–Casher effect [8]. Assume this force exists and show how to violate energy conservation by placing reflectors in the path of the current loop.

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14 Quantum Measurements and Relativity

Previous chapters discuss nonrelativistic quantum mechanics. Is there a relativistic quantum mechanics? In this chapter we assume that there is – and arrive at a paradox. The paradox concerns Lorentz transformations of quantum measurements. At the end of a quantum measurement, an entangled state of the measured system and measuring device collapses to a product state, according to von Neumann. (See Chap. 9.) The collapse is not Lorentz invariant, so we try to make it Lorentz invariant. The paradox is that there seems to be *no way* to make collapse Lorentz invariant.

The paradox concerns the collapse, not the correlations, of entangled states. If Alice and Bob make local measurements, their results are Lorentz invariant, and the correlations in their results are Lorentz invariant, as well. Bell's inequality and Eq. (3.12) do not depend on how we resolve the paradox of Sect. 14.1; nonlocal quantum correlations are nonlocal in any inertial reference frame.

14.1 Collapse and Relativity

In 1927, at the fifth Solvay congress, Einstein presented "a very simple objection to the probability interpretation" of quantum mechanics [1]. Consider a beam of electrons incident on a screen with a single slit. Some of the electrons pass through the slit and form a diffraction pattern on a photographic plate on the other side. According to quantum mechanics, the state of the electrons approaching the photographic plate is an extended object, and the probability density for an electron to hit varies smoothly over the plate. Once the electron hits the plate, however, the probability for the electron to hit anywhere else on the plate drops to zero. Apparently, then, the state changes instantaneously when the electron hits the plate. But the instantaneous collapse of an extended object is incompatible with relativity. So how can the electron state collapse instantaneously?

Similarly, suppose we measure the momentum of a particle very carefully, taking as much time as we need for the measurement. By the end of the measurement, the particle is in a momentum eigenstate. At time $t = 0$ we turn on a particle detector at the origin and find the particle there. According to von Neumann's collapse postulate, the state of the particle changes instantaneously, along the $t = 0$ hyperplane, from a momentum eigenstate to a position eigenstate. (See Fig. 14.1 and Sect. 9.1.) The problem is that collapse along the $t = 0$ hyperplane will not be instantaneous for observers in other frames. Thus Fig. 14.2, the Lorentz transformation of Fig. 14.1, makes no sense: the detector has found the particle at the origin, yet the wave function of the particle does not vanish everywhere else.

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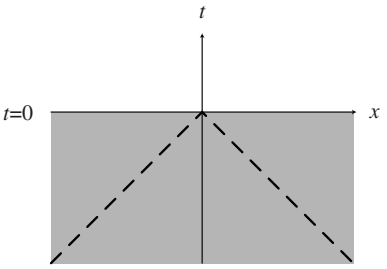

Figure 14.1: A position measurement at $t = 0$, with collapse on the $t = 0$ hyperplane. The particle is in a momentum eigenstate until $t = 0$.

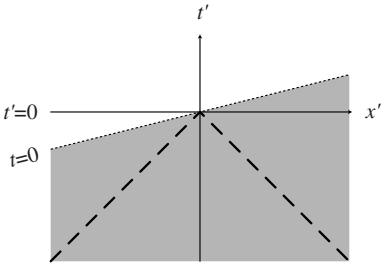

Figure 14.2: The collapse of Fig. 14.1 in a different frame.

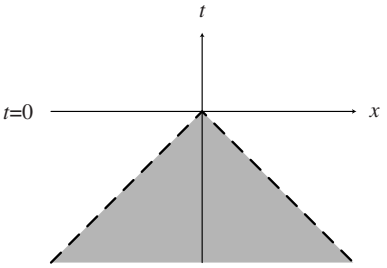

Figure 14.3: Collapse along the past light cone of a measurement.

Indeed, even Fig. 14.1 makes no sense. For according to Fig. 14.1, the wave function of the particle did not vanish anywhere until $t = 0$. But since the particle cannot move faster than light, it could not have been outside the past light cone of its detection at the origin at $t = 0$; the wave function should vanish outside the past light cone. Could collapse perhaps occur along the past light cone, as in Fig. 14.3? Collapse along the past light cone [2] would be Lorentz invariant because light cones are Lorentz invariant. But for $t < 0$, Fig. 14.3 is not compatible with any momentum eigenstate.

Another version of this paradox involves two distinguishable particles in an entangled state, the singlet state $|\Psi_-\rangle$:

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \ . \tag{14.1}$$

(In Eq. (14.1), as in Eqs. (3.5–6), we let the order of the states in each tensor product index the particles.) Equation (14.1) represents only the spin states of the particles, not their spatial wave functions, but we assume that the two particles are localized to regions that are far from each other. Figure 14.4 shows the spacetime history of the two particles. They emerge from O in the entangled state $|\Psi_-\rangle$ and fly off at nearly the speed of light in opposite directions.

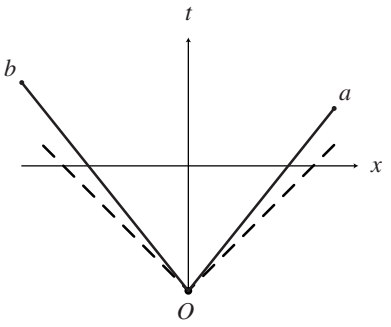

Figure 14.4: Measurements at a and b on particles that left O in an entangled state.

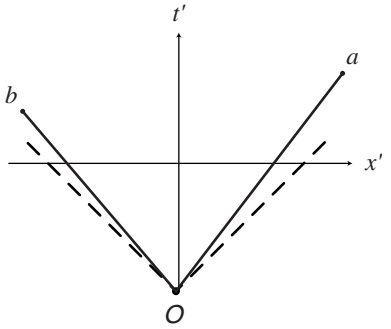

Figure 14.5: The events of Fig. 14.4 in a different frame.

One particle reaches Alice, who (at a) measures its spin component along some axis, and the other particle reaches Bob, who (at b) measures its spin component along a (possibly different) axis. Alice's measurement is spacelike separated from Bob's. In Fig. 14.4, a is earlier than b, so by the time Bob makes a measurement, his particle is no longer entangled with Alice's; from the result of her measurement, Alice can anticipate the spin state of Bob's particle. So according to Fig 14.4, Alice's measurement collapses the entangled state of the two particles. But to an observer in another inertial reference frame, Bob's measurement could come earlier. Figure 14.5 shows the spacetime diagram for such an observer; the same events appear, but in Fig. 14.5, b precedes a. For this observer, Bob's measurement – not Alice's – collapses the entangled state. The two observers cannot agree on who collapsed the state. Even if we assume that collapse occurs along the past light cone of a measurement, we cannot say whether it was Alice's or Bob's measurement.

Let us fix a and put b at any point spacelike separated from a. In Fig. 14.5, b precedes a, but the time order of a and b is relative since their separation is spacelike. As long as b is outside the past light cone of a, there is always a frame in which a precedes b. There is a frame in which Alice has already measured when Bob makes his measurements. Thus the collapse cannot occur outside the past light cone of a. Hence it cannot occur outside the past light cone of b either. It seems that collapse has *not* occurred only in the intersection of the past light cones of a and b. Perhaps, then, collapse occurs on the (Lorentz invariant) boundary of this intersection. (See Fig. 14.6.) If so, then the particles are in a *product* state when Alice and Bob make their measurements. But now suppose many pairs of particles emerge from O in the state $|\Psi_-\rangle$. As in Fig. 14.4, one particle in each pair goes to Alice, the other to Bob. At

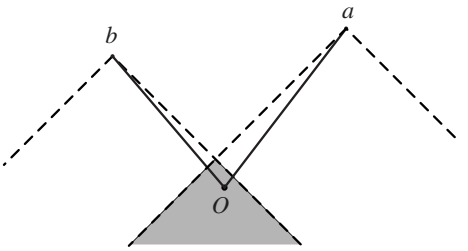

Figure 14.6: Collapse on the boundary of the intersection of the past light cones of a and b.

a, Alice measures the spin components of all her particles, along various axes. Bob does the same with his particles at b. If the pairs of particles are in a product state when they reach a and b, the correlations in these measurements must be local, i.e. they must satisfy Bell's inequality. (See Sect. 3.4.) But the quantum correlations of $|\Psi_-\rangle$ do not satisfy Bell's inequality. So the particles cannot be in a product state when Alice and Bob make their measurements.

14.2 Relativistic Constraints on Measurements

In 1931, Landau and Peierls [3] claimed that the theory of relativity imposes constraints on quantum measurements, beyond the constraints of the Heisenberg uncertainty relations.[1] They based their claim on a paradox. Suppose that at an initial time $t < 0$, we localize a particle in a small region of space; then at $t = 0$, we measure its momentum. Our measurement leaves the particle in some eigenstate of momentum. For any eigenstate of momentum, the probability density in space is uniform. So – whatever the result of our measurement – if we look for the particle again at any time $t > 0$, we may find it anywhere. We may find it at a spacelike separation from its initial position. Thus quantum mechanics contradicts relativity theory. Landau and Peierls resolved this paradox by claiming that a measurement of momentum cannot be instantaneous: a momentum measurement of accuracy Δp cannot take time less than Δt, where Δt satisfies a relativistic uncertainty relation

$$\Delta p \Delta t \geq \hbar/c \,. \tag{14.2}$$

Equation (14.2) follows from Heisenberg's uncertainty relation $\Delta p \Delta x \geq \hbar$ if we assume $\Delta x \leq c\Delta t$. More generally, Landau and Peierls claimed that a measurement of any nonlocal operator on a system cannot be instantaneous.

Let us apply their claim to a measurement of the total spin of two localized spin-1/2 particles that are remote from one another [4]. As before, our friends Alice and Bob help us with the measurement. Alice and Bob have prepared two spin-1/2 particles in the spin state

$$|\Psi_{\alpha\beta}\rangle = \alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle \,, \tag{14.3}$$

with $|\alpha|^2 + |\beta|^2 = 1$. Alice and Bob each take one of the particles; in the tensor products of Eq. (14.3), the first spin state refers to Alice's particle and the second to Bob's. Let $\mathbf{S}^{(A)}$ and $\mathbf{S}^{(B)}$ represent the spins of Alice's and Bob's particles, respectively. Let us assume that

[1] They also claimed that a measurement of the electric field cannot be impulsive. See Chap. 8.

Alice and Bob can measure the operator S^2 on $|\Psi_{\alpha\beta}\rangle$, where $\mathbf{S} = \mathbf{S}^{(A)} + \mathbf{S}^{(B)}$, and that their measurements are impulsive and simultaneous (in some inertial reference frame). According to the claim of Landau and Peierls, such a measurement contradicts relativistic causality. Does it?

In general, $|\Psi_{\alpha\beta}\rangle$ is not an eigenstate of S^2, but $|\Psi_{\alpha\beta}\rangle$ can be written as a sum of eigenstates of S^2:

$$|\Psi_{\alpha\beta}\rangle = \frac{\alpha+\beta}{\sqrt{2}}|2,0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|0,0\rangle \ ,$$

where

$$|2,0\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

is the eigenstate of S^2 with $S^2 = 2\hbar^2$ and $S_z \equiv S_z^{(A)} + S_z^{(B)} = 0$, and

$$|0,0\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

is the eigenstate of S^2 with $S^2 = 0 = S_z$. Thus a measurement of S^2 on $|\Psi_{\alpha\beta}\rangle$ must yield $2\hbar^2$ with probability $|\alpha+\beta|^2/2$ and 0 with probability $|\alpha-\beta|^2/2$.

A special case of $|\Psi_{\alpha\beta}\rangle$ is the state $|\uparrow\downarrow\rangle$, i.e. $\beta = 0$. Suppose that the two systems are initially in the state $|\uparrow\downarrow\rangle$, and consider two possible sequences of measurements on this state. In one sequence, Alice and Bob measure S^2 on the particles at time t_1. The initial state is a superposition of eigenstates of S^2

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle + |0,0\rangle) \ .$$

A measurement of S^2 must leave the particles either in the state $|2,0\rangle$ or in the state $|0,0\rangle$, with equal probability. Now at a time $t_2 > t_1$ Alice measures the operator $S_z^{(A)}$ on her system. No matter what the result of the S^2 measurement, there is an equal probability for Alice to find $S_z^{(A)} = \hbar/2$ and $S_z^{(A)} = -\hbar/2$.

The second sequence of measurements is the same as the first, except that just before t_1 – just before the measurement of S^2 – Bob flips the spin of his particle from $|\downarrow\rangle_B$ to $|\uparrow\rangle_B$. (The subscript B refers to Bob.) He thereby changes the state of the two particles from $|\uparrow\downarrow\rangle$ to $|\uparrow\uparrow\rangle$. The state $|\uparrow\uparrow\rangle$ is an eigenstate $|2,1\rangle$ of S^2 with eigenvalue $2\hbar^2$ and of S_z with eigenvalue $\hbar$. The measurement of S^2 at $t = t_1$ therefore yields $2\hbar^2$. As in the sequence above, at $t_2 > t_1$ Alice measures the operator $S_z^{(A)}$ on her particle. But now Alice obtains the result $S_z^{(A)} = \hbar/2$ with certainty! She is certain to obtain $S_z^{(A)} = \hbar/2$ just because Bob flipped the spin of his particle before t_1. By assumption, the time interval from t_1 to t_2 can be arbitrarily short (since the S^2 measurement is impulsive). Thus Bob's local measurement can affect the result of Alice's measurement over a spacelike distance, in violation of relativistic causality. Bob could transmit a superluminal signal to Alice if they repeated the experiment on many pairs of spin. If Bob flips none of his spins, Alice obtains $S_z^{(A)} = -\hbar/2$ and $S_z^{(A)} = \hbar/2$ with equal probability; if Bob flips all his spins, Alice obtains $S_z^{(A)} = \hbar/2$ only.

We arrive at this violation of relativistic causality by assuming that a measurement of S^2 on the state $|\uparrow\downarrow\rangle$ could be practically instantaneous. So Landau and Peierls were right when they claimed that an instantaneous measurement of certain nonlocal operators would violate relativistic causality. However, we have not shown that the instantaneous measurement of *every* nonlocal operator on *every* state would violate relativistic causality. Let us consider a more general measurement. We return to the entangled state $|\Psi_{\alpha\beta}\rangle$ in Eq. (14.3) – without assuming $\beta = 0$ – and apply the two measurement sequences to this state. In the first sequence, Alice and Bob measure S^2 and then Alice measures $S_z^{(A)}$ on her particle. The outcome of the S^2 measurement leaves the particles either in the state $|0,0\rangle$ or in the state $|2,0\rangle$. In either $|0,0\rangle$ or $|2,0\rangle$, the results $S_z^{(A)} = \pm\hbar/2$ for Alice's measurement are equally likely. The second sequence is the same as the first, except that Bob first flips the spin of his particle. He thereby changes the state from $|\Psi_{\alpha\beta}\rangle$ to

$$|\Psi'_{\alpha\beta}\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\downarrow\downarrow\rangle \ ,$$

and $|\Psi'_{\alpha\beta}\rangle$ is an eigenstate of S^2 with eigenvalue $2\hbar^2$. Hence the S^2 measurement does not change the state of the two particles. Now if Alice measures $S_z^{(A)}$ on her particle, the probability of the result $S_z^{(A)} = \hbar/2$ is $|\alpha|^2$ while the probability of the result $S_z^{(A)} = -\hbar/2$ is $|\beta|^2$. So if $|\alpha|^2 \neq |\beta|^2$, such sequences of measurements would allow Alice and Bob to violate causality.

This analysis shows that an instantaneous measurement of S^2 on any state $|\Psi_{\alpha\beta}\rangle$ with $|\alpha|^2 \neq |\beta|^2$ would be incompatible with relativistic causality. While it supports the general claim of Landau and Peierls, it also suggests a loophole: perhaps a measurement of S^2 on $|\Psi_{\alpha\beta}\rangle$ could be instantaneous, *only* in the case $|\alpha|^2 = |\beta|^2$. The next section shows that this suggestion is correct.

14.3 Nonlocal Measurements

In this section we encounter a new kind of measurement. We will see how Alice and Bob can verify that their spin-1/2 particles are in the singlet state $|0,0\rangle$, with an instantaneous measurement [5]. The state $|0,0\rangle$ is the eigenstate of S^2 with eigenvalue 0, so we will see how Alice and Bob can measure S^2 on this state, although – as the last section shows – they *cannot* measure S^2 on the state $|\uparrow\downarrow\rangle$. Thus Alice and Bob find themselves verifying a state instead of measuring an operator; unlike nonrelativistic quantum mechanics, which allows an instantaneous measurement of any observable on any state, relativistic causality allows an instantaneous measurement of S^2 on some states but not on others!

To construct a quantum measurement that verifies $S^2 = 0$, we apply the formalism of Chap. 7. The first step in the construction is to show that Alice and Bob can measure S_z without measuring either $S_z^{(A)}$ or $S_z^{(B)}$. They do so with a Hamiltonian

$$H_{int} = g(t)\left[S_z^{(A)}P_z^A + S_z^{(B)}P_z^B\right] \ , \tag{14.4}$$

after preparing their measuring devices in a state with

$$Q_z^A + Q_z^B = 0 = P_z^A - P_z^B \ . \tag{14.5}$$

Here Q_z^A and Q_z^B represent the positions of pointers on the measuring devices of Alice and Bob, respectively; P_z^A and P_z^B are conjugate to Q_z^A and Q_z^B, respectively. The coupling $g(t)$ is zero except during the interval $0 \leq t \leq T$, when $g(t) = 1/T$; we assume that T approaches 0, so the measurement is impulsive. The twist in this measurement is that Alice and Bob prepare their measuring devices in an entangled state, Eq. (14.5); it is, in fact, the entangled state of Einstein, Podolsky and Rosen. (See Sect. 3.1.) Now applying the Heisenberg equations of motion, we find that P_z^A and P_z^B do not change during the measurement, while $Q_z^A + Q_z^B$ changes from 0 to the value $S_z^{(A)} + S_z^{(B)}$. So Alice and Bob have measured S_z on their entangled state. But they have not measured $S_z^{(A)}$ or $S_z^{(B)}$, because neither Q_z^A nor Q_z^B is defined initially.

How much time does this procedure take? It takes some time to prepare the measuring devices in the state of Eq. (14.5) and to bring them to where Alice and Bob use them. After the impulsive measurement, it again takes some time for Alice and Bob to meet and compare their results (the final values $Q_z^A(T)$ and $Q_z^B(T)$ that they have recorded in their notebooks). But the interaction between the measuring device and the entangled particles is impulsive; the measurement is instantaneous, because it fixes the result immediately. Alice and Bob read the result only later, but the measurement is over when the interaction is over. Thus – contrary to the claim of Landau and Peierls – there can indeed be instantaneous measurements of nonlocal properties.

Now the state $|0,0\rangle$ is the unique state for which the operators S_x, S_y and S_z all vanish. Thus, if Alice and Bob measure these three operators on a pair of particles and obtain 0 for all three, they have verified that the state is $|0,0\rangle$. Such a measurement is plausible, because Alice and Bob can measure S_x, S_y and S_z separately and instantaneously, as we have just seen. An interaction Hamiltonian that includes these three measurements is

$$H_{int} = g(t) \left[S_x^{(A)} P_x^A + S_x^{(B)} P_x^B + S_y^{(A)} P_y^A + S_y^{(B)} P_y^B + S_z^{(A)} P_z^A + S_z^{(B)} P_z^B \right] . \tag{14.6}$$

Alice and Bob prepare their measuring devices in an initial state

$$\begin{aligned} Q_x^A + Q_x^B = 0 = P_x^A - P_x^B \ , \\ Q_y^A + Q_y^B = 0 = P_y^A - P_y^B \ , \\ Q_y^A + Q_y^B = 0 = P_y^A - P_y^B \ . \end{aligned} \tag{14.7}$$

We must check the evolution of 18 variables, namely six pointer positions and their conjugate momenta, and six spin components. Despite the large number of variables, Eqs. (14.6–7) do not present greater difficulty than do Eqs. (14.4–5). The conjugate momenta do not change, since they commute with H_{int}. The evolution links pointer positions and spin components:

$$\begin{aligned} Q_x^A(T) + Q_x^B(T) - Q_x^A(0) - Q_x^B(0) = S_x^{(A)} + S_x^{(B)} = 0 \ , \\ Q_y^A(T) + Q_y^B(T) - Q_y^A(0) - Q_y^B(0) = S_y^{(A)} + S_y^{(B)} = 0 \ , \\ Q_z^A(T) + Q_z^B(T) - Q_z^A(0) - Q_z^B(0) = S_z^{(A)} + S_z^{(B)} = 0 \ , \end{aligned} \tag{14.8}$$

while

$$\begin{aligned}
&S_x^{(A)}(T) + S_x^{(B)}(T) - S_x^{(A)}(0) - S_x^{(B)}(0) \\
&\quad = S_z^{(A)} P_y^A + S_z^{(B)} P_y^B - S_y^{(A)} P_z^A - S_y^{(B)} P_z^B = 0 \\
&S_y^{(A)}(T) + S_y^{(B)}(T) - S_y^{(A)}(0) - S_y^{(B)}(0) \\
&\quad = S_x^{(A)} P_z^A + S_x^{(B)} P_z^B - S_z^{(A)} P_x^A - S_z^{(B)} P_x^B = 0 \\
&S_z^{(A)}(T) + S_z^{(B)}(T) - S_z^{(A)}(0) - S_z^{(B)}(0) \\
&\quad = S_y^{(A)} P_x^A + S_y^{(B)} P_x^B - S_y^{(A)} P_y^A - S_y^{(B)} P_y^B = 0 \, .
\end{aligned} \tag{14.9}$$

We have used the fact that $S_x^{(A)} + S_x^{(B)} = S_y^{(A)} + S_y^{(B)} = S_z^{(A)} + S_z^{(B)} = 0$ for the singlet state $|0,0\rangle$; otherwise, the evolution would be more complicated.

Thus the procedure for verifying the entangled state $|0,0\rangle$ is the following: at a time $t = 0$ the measuring devices, in the entangled state of Eq. (14.7), are ready and in place in the laboratories of Alice and Bob. Between times $t = 0$ and $t = T$, the coupling $g(t)$ and the Hamiltonian H_{int} of Eq. (14.6) are nonzero. At time $t = T$ the interaction is complete, and Alice and Bob check and record the pointer positions on their measuring devices. It still takes them time to compare their records, but the measurement is complete. If Alice and Bob obtain $Q_x^A + Q_x^B = Q_y^A + Q_y^B = Q_y^A + Q_y^B = 0$ *after* the measurement, they have verified that $S_x = S_y = S_z = S^2 = 0$.

14.4 Which Nonlocal Operators are Measurable?

The difference between local and nonlocal measurements of S^2 is striking. If Alice and Bob bring their particles together into one laboratory, they can measure the operator S^2 on any spin state; we can call their measurement an *operator-specific* measurement. By contrast, when Alice and Bob (with their respective laboratories and particles) are far from each other, they can at best verify eigenstates of S^2. The previous section shows how Alice and Bob can verify the state $|0,0\rangle$, in what we can call a *state-specific* measurement; a modification of this measurement would allow them to verify the state $|2,0\rangle$. They could also verify the states $|2,1\rangle$ and $|2,-1\rangle$ by local measurements of $S_z^{(A)}$ and $S_z^{(B)}$, but these local measurements are incompatible with nonlocal measurements to verify $|0,0\rangle$ and $|2,0\rangle$. Thus Alice and Bob can verify any eigenstate of S^2, yet they cannot measure S^2 in an instantaneous operator-specific measurement. The distinction between operator-specific and state-specific measurements arises only when we consider relativistic causality.

Now we ask, is there *any* nonlocal operator – any operator with entangled eigenstates – that relativistic causality allows Alice and Bob to measure instantaneously? This section shows that almost any such measurement would contradict relativistic causality. But (like Sect. 14.2) it suggests a loophole; and as the next section shows, relativistic causality *does* allow Alice and Bob to measure at least *one* nonlocal operator instantaneously [6].

As before, we consider spin operators. Consider pairs of distinguishable spin-1/2 particles, with one particle of each pair in Alice's laboratory and the other in Bob's laboratory. Let W be a nondegenerate operator on the Hilbert space of the two spins, and let W have, as one of

its eigenstates, the spin state $|\Psi_{\alpha\beta}\rangle$ of Eq. (14.3):

$$|\Psi_{\alpha\beta}\rangle = \alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle \ .$$

For simplicity, we take α and β real. Can Alice and Bob instantaneously measure W on their spins? Let us suppose they can. Assume that Alice and Bob can measure W by means of measuring devices that interact impulsively (and locally) with the spins, such that if the spins are in the initial state $|\Psi_{\alpha\beta}\rangle$, the measuring devices verify this state without changing it, and record the result; if the initial state of the spins is orthogonal to $|\Psi_{\alpha\beta}\rangle$, the measuring devices record a different result, and leave the spins in an eigenstate orthogonal to $|\Psi_{\alpha\beta}\rangle$. (This assumption holds for quantum measurements of any local observable.) The operator W has three additional eigenstates, which we denote $|W_1\rangle$, $|W_2\rangle$, $|W_3\rangle$. They span the subspace orthogonal to $|\Psi_{\alpha\beta}\rangle$. We need not define $|W_1\rangle$, $|W_2\rangle$ and $|W_3\rangle$, but we define three orthonormal vectors that span the same subspace: they are $|\Psi^{\perp}_{\alpha\beta}\rangle$, $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, with

$$|\Psi^{\perp}_{\alpha\beta}\rangle = \beta|\uparrow\downarrow\rangle - \alpha|\downarrow\uparrow\rangle \ .$$

Now Alice and Bob try the following experiment. First, they prepare an ensemble of pairs in the singlet state $|0,0\rangle$:

$$\begin{aligned} |0,0\rangle &= \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right] \\ &= \frac{\alpha-\beta}{\sqrt{2}}|\Psi_{\alpha\beta}\rangle + \frac{\alpha+\beta}{\sqrt{2}}|\Psi^{\perp}_{\alpha\beta}\rangle \ . \end{aligned}$$

Second, they measure W instantaneously on each pair of spins. With probability $(\alpha-\beta)^2/2 = (1-2\alpha\beta)/2$, their measuring devices record a result corresponding to the state $|\Psi_{\alpha\beta}\rangle$, and leave the pair of spins in the state $|\Psi_{\alpha\beta}\rangle$. However, the measuring devices may record a result corresponding to another eigenstate $|W_i\rangle$, and leave the pair of spins in the state $|W_i\rangle$. The probability of this result is

$$\frac{(\alpha+\beta)^2}{2}\left|\langle\Psi^{\perp}_{\alpha\beta}|W_i\rangle\right|^2 = \frac{1+2\alpha\beta}{2}\left|\langle\Psi^{\perp}_{\alpha\beta}|W_i\rangle\right|^2 \ .$$

Finally, Alice measures $S_z^{(A)}$ and Bob measures $S_z^{(B)}$.

What is the probability that Alice obtains $S_z^{(A)} = \hbar/2$? If the pair is in the state $|\Psi_{\alpha\beta}\rangle$, the probability is $|\langle\Psi_{\alpha\beta}|\uparrow\rangle_A|^2 = \alpha^2$, where $|\uparrow\rangle_A$ represents the state of Alice's spin with $S_z^{(A)} = \hbar/2$. What if the pair is in the state $|W_i\rangle$? Then the probability that Alice obtains $S_z^{(A)} = \hbar/2$ is $|\langle W_i|\uparrow\rangle_A|^2$. Since

$$\langle W_i| = \langle W_i|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \langle W_i|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + \langle W_i|\Psi^{\perp}_{\alpha\beta}\rangle\langle\Psi^{\perp}_{\alpha\beta}| \ ,$$

we have

$$\begin{aligned} |\langle W_i|\uparrow\rangle_A|^2 &= |\langle W_i|\uparrow\uparrow\rangle|^2|\langle\uparrow\uparrow|\uparrow\rangle_A|^2 + |\langle W_i|\Psi^{\perp}_{\alpha\beta}\rangle|^2|\langle\Psi^{\perp}_{\alpha\beta}|\uparrow\rangle_A|^2 \\ &= |\langle W_i|\uparrow\uparrow\rangle|^2 + |\langle W_i|\Psi^{\perp}_{\alpha\beta}\rangle|^2\beta^2 \ . \end{aligned}$$

Thus, the probability $P(\uparrow_A)$ that Alice obtains $S_z^{(A)} = \hbar/2$ after the measurement of W is

$$P(\uparrow_A) = \frac{1-2\alpha\beta}{2}\alpha^2 + \frac{1+2\alpha\beta}{2}\sum_i |\langle W_i|\Psi^\perp_{\alpha\beta}\rangle|^2 \left[|\langle W_i|\uparrow\uparrow\rangle|^2 + |\langle W_i|\Psi^\perp_{\alpha\beta}\rangle|^2\beta^2\right] . \quad (14.10)$$

Similarly, the probability $P(\uparrow_B)$ that Bob obtains $S_z^{(B)} = \hbar/2$ is

$$P(\uparrow_B) = \frac{1-2\alpha\beta}{2}\beta^2 + \frac{1+2\alpha\beta}{2}\sum_i |\langle W_i|\Psi^\perp_{\alpha\beta}\rangle|^2 \left[|\langle W_i|\uparrow\uparrow\rangle|^2 + |\langle W_i|\Psi^\perp_{\alpha\beta}\rangle|^2\alpha^2\right] . \quad (14.11)$$

Here relativistic causality imposes a constraint. *After* Alice and Bob prepare the ensemble of pairs in the state $|0,0\rangle$ but *before* they measure W, either Alice or Bob could change the state of the pairs from $|0,0\rangle$ to $|2,0\rangle$. For example, by briefly subjecting her spins to a magnetic field in the z-direction, Alice could change the relative phase between the terms $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ in $|0,0\rangle$ as she likes, in an arbitrarily short time. So could Bob. These actions belong to a general class of local operations, *local unitary transformations*. Local unitary transformations are unitary transformations that apply *only* to Alice's system or *only* to Bob's (or products of these). If Alice can affect $P(\uparrow_B)$ or Bob can affect $P(\uparrow_A)$ by such local actions, one can send a superluminal signal to the other. So relativistic causality implies that if Alice and Bob prepare their spins in the state $|2,0\rangle$,

$$\begin{aligned}|2,0\rangle &= \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right] \\ &= \frac{\alpha+\beta}{\sqrt{2}}|\Psi_{\alpha\beta}\rangle - \frac{\alpha-\beta}{\sqrt{2}}|\Psi^\perp_{\alpha\beta}\rangle ,\end{aligned}$$

measure W, and measure $S_z^{(A)}$ and $S_z^{(B)}$, the probabilities $P(\uparrow_A)$ and $P(\uparrow_B)$ must be the *same* as in Eqs. (14.10–11). So besides Eqs. (14.10–11) we have

$$\begin{aligned}P(\uparrow_A) &= \frac{1+2\alpha\beta}{2}\alpha^2 + \frac{1-2\alpha\beta}{2}\sum_i |\langle W_i|\Psi^\perp_{\alpha\beta}\rangle|^2 \left[|\langle W_i|\uparrow\uparrow\rangle|^2 + |\langle W_i|\Psi^\perp_{\alpha\beta}\rangle|^2\beta^2\right] , \\ P(\uparrow_B) &= \frac{1+2\alpha\beta}{2}\beta^2 + \frac{1-2\alpha\beta}{2}\sum_i |\langle W_i|\Psi^\perp_{\alpha\beta}\rangle|^2 \left[|\langle W_i|\uparrow\uparrow\rangle|^2 + |\langle W_i|\Psi^\perp_{\alpha\beta}\rangle|^2\alpha^2\right] . \quad (14.12)\end{aligned}$$

Equating $P(\uparrow_A) - P(\uparrow_B)$ across Eqs. (14.10–11) and Eq. (14.12), we find that $\alpha\beta = 0$ or $\alpha = \pm\beta$.

This conclusion holds for any nondegenerate operator W, on the Hilbert space of the two spins, having $|\Psi_{\alpha\beta}\rangle$ as an eigenstate. But actually, local unitary transformations can reduce any state in the Hilbert space of the two spins to the state $|\Psi_{\alpha\beta}\rangle$ for some α and β real. (See

Prob. 14.4.) Just as Alice and Bob can change a relative phase by local actions, Alice can flip or rotate her spin (without knowing what it is), or Bob can flip his; these actions, too, are local unitary transformations. If Alice and Bob share a pair of spins in any state, they can reduce the state to the form $|\Psi_{\alpha\beta}\rangle$ by local unitary operations. They cannot, however, change $|\alpha|$ and $|\beta|$. Thus relativistic causality imposes the following condition: Alice and Bob cannot measure W instantaneously unless each eigenstate of W equals $|\Psi_{\alpha\beta}\rangle$ – for some α and β satisfying $\alpha\beta = 0$ or $|\alpha| = |\beta|$ – up to local unitary transformations.

This condition is necessary but not sufficient; S^2 satisfies it, yet Alice and Bob cannot measure S^2 (or any nondegenerate operator with the same eigenstates) instantaneously. Does W exist?

14.5 Measuring a Nonlocal Operator

Can Alice and Bob measure even *one* nonlocal operator instantaneously? Actually, they can. To show how they can, we first consider a variation of the nonlocal measurements of Sect. 14.3.

Suppose Alice and Bob share a pair of spin-1/2 particles and wish to determine *not* $S_z^{(A)} + S_z^{(B)}$ but $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$. If the state of their spins is $|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$, they can simply measure $S_z^{(A)} + S_z^{(B)}$ to determine $(S_z^{(A)} + S_z^{(B)}) \bmod 2\hbar$. But what if the state of their spins is $(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$? This state is an eigenstate of $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ but not of $S_z^{(A)} + S_z^{(B)}$. Alice and Bob cannot determine $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ by measuring $S_z^{(A)} + S_z^{(B)}$. They have to measure $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ directly. Can they?

Indeed, Alice and Bob can measure $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ and $S_z^{(A)} + S_z^{(B)}$ with the *same* Hamiltonian, H_{int}, of Eq. (14.4). But the initial (entangled) state of their measuring devices must be different. To measure $S_z^{(A)} + S_z^{(B)}$, Alice and Bob prepare their measuring devices in the state $Q_z^A + Q_z^B = 0$. (Here, as in Sect. 14.3, Q_z^A and Q_z^B represent pointer positions, P_z^A and P_z^B their respective conjugate momenta.) To measure $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$, Alice and Bob must prepare their measuring devices in the state $Q_z^A + Q_z^B = 0 \bmod 2\hbar$. As in Sect. 14.3, H_{int} evolves $Q_z^A(0) + Q_z^B(0)$ to

$$Q_z^A(T) + Q_z^B(T) = Q_z^A(0) + Q_z^B(0) + S_z^{(A)} + S_z^{(B)} \ ,$$

but now a final measurement of $Q_z^A(T) + Q_z^B(T)$ fixes $S_z^{(A)} + S_z^{(B)}$ only up to a multiple of $2\hbar$, so Alice and Bob have measured $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ but not $S_z^{(A)} + S_z^{(B)}$.

How does this measurement affect the eigenstates of $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$? We want a measurement of $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ on one of its eigenstates to leave the eigenstate unchanged (up to an overall phase). Since

$$e^{-i\int H_{int}dt/\hbar} = e^{-i\left[P_z^A S_z^{(A)} + P_z^B S_z^{(B)}\right]/\hbar} \ ,$$

we have

$$e^{-i\int H_{int}dt/\hbar}\,(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle) = e^{-i(P_z^A - P_z^B)/2}|\uparrow\downarrow\rangle \pm e^{i(P_z^A - P_z^B)/2}|\downarrow\uparrow\rangle \ ,$$
$$e^{-i\int H_{int}dt/\hbar}\,(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle) = e^{-i(P_z^A + P_z^B)/2}|\uparrow\uparrow\rangle \pm e^{i(P_z^A + P_z^B)/2}|\downarrow\downarrow\rangle \ .$$

So the requirement that the measurement not change the eigenstates implies

$$e^{i(P_z^A - P_z^B)} = 1 = e^{i(P_z^A + P_z^B)} ,$$

i.e. P_z^A and P_z^B must be multiples of π and $P_z^A - P_z^B$ must be a multiple of 2π. This constraint is compatible with the assumption $Q_z^A + Q_z^B = 0 \bmod 2\hbar$, hence Alice and Bob can measure $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ instantaneously. (See Prob. 14.6.)

Now consider an operator W with eigenstates $|0,0\rangle$, $|2,0\rangle$, $(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ and $(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)/\sqrt{2}$. Alice and Bob can measure W instantaneously in four steps. In the first step, Alice and Bob measure $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ on their spins. This measurement distinguishes $|0,0\rangle$ and $|2,0\rangle$, on the one hand, from $(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ and $(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)/\sqrt{2}$, on the other. Next, they must distinguish $|0,0\rangle$ from $|2,0\rangle$ or $(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ from $(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)/\sqrt{2}$. To do so, Alice rotates her spin as follows:

$$|\uparrow\rangle_A \to (|\uparrow\rangle_A - |\downarrow\rangle_A)/\sqrt{2} , \quad |\downarrow\rangle_A \to (|\uparrow\rangle_A + |\downarrow\rangle_A)/\sqrt{2} . \tag{14.13}$$

Bob does the same to his spin. Equation (14.13) defines a local unitary operation. By applying Eq. (14.13) to their respective spins, Alice and Bob interchange $|2,0\rangle$ and $(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)/\sqrt{2}$ while leaving $|0,0\rangle$ and $(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ unchanged. In the third step, Alice and Bob measure $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ again. From the results of the two $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ measurements they can positively identify each of the four eigenstates $|0,0\rangle$, $|2,0\rangle$, $(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ and $(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)/\sqrt{2}$ of W. Finally, Alice and Bob apply the inverse of Eq. (14.13) so that the measurement of W returns the spins to their initial state.

Indeed, W is a nonlocal operator that Alice and Bob can measure. Each eigenstate of W is equivalent (up to local unitary transformations) to $|\Psi_{\alpha\beta}\rangle$ with $\alpha = \pm\beta$; we say it is maximally entangled [7]. What about an operator with an eigenstate that is not entangled ($\alpha\beta = 0$)? Can Alice and Bob instantaneously measure such an operator? A nonlocal operator must have $\alpha\beta \neq 0$ for at least one eigenstate. So consider an operator having at least one maximally entangled eigenstate and at least one product eigenstate. We can take these two eigenstates to be[2] $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle/\sqrt{2}$ and $|\uparrow\uparrow\rangle$. Could Alice and Bob instantaneously measure an operator with these two states as eigenstates? If they could, they could also do the following. Let Alice and Bob prepare their spins in the state $|\uparrow\uparrow\rangle$. Next, they measure the operator instantaneously. Finally, Alice measures σ_z on her spin. She is certain to obtain $\sigma_z = 1$. But what if, just before they measure the nonlocal operator, Bob flips his spin? He changes the initial state from $|\uparrow\uparrow\rangle$ to $|\uparrow\downarrow\rangle$, and now the measurement of the nonlocal operator has a chance of leaving the spins in the state $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ and Alice has a chance of obtaining -1 when she measures σ_z on her spin. If so, Bob could send a superluminal signal to Alice. We conclude that Alice and Bob *cannot* measure such an operator instantaneously.

To summarize, Alice and Bob can measure W instantaneously (although they cannot obtain the result of their measurement instantaneously). However, W is essentially the *only* nondegenerate nonlocal operator that Alice and Bob can measure instantaneously on their two spins.

[2]First, any maximally entangled spin state of the two particles is equivalent, up to local unitary transformations,local unitary transformations to the singlet state $|0,0\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. (See Prob. 14.4.) Second, the singlet state is invariant under any overall rotation – a rotation of Alice's spin, and the same rotation of Bob's spin – hence Alice and Bob can bring the product eigenstate to the form $|\uparrow\uparrow'\rangle$ where $|\uparrow'\rangle_B$ represents Bob's spin polarized along some axis, not necessarily the same as Alice's z-axis. But nondegenerate eigenstates must be orthogonal. Hence $|\uparrow'\rangle_B = |\uparrow\rangle_B$.

That is, Alice and Bob can measure only those operators related to W by local unitary transformations. To measure any such operator – any nondegenerate operator with four maximally entangled eigenstates – they can apply the local unitary transformations that transform it to W, measure W, and then apply the inverse transformations. The proof that suitable local unitary transformations exist, for every such operator, is Prob. 14.8.

14.6 Collapse and Relativity Revisited

The paradoxes of Sect. 14.1 suggest that there is no Lorentz invariant account of collapse, and the nonlocal measurements of Sects. 14.2–14.5 reinforce this suggestion. Suppose Alice and Bob prepare a pair of distinguishable spin-1/2 particles in the entangled state $|0,0\rangle$. Each has a particle and, at time $t = 0$, measures its spin along some axis. Figure 14.7 shows Alice's measurement at a and Bob's measurement at b. In addition, *before* a and b, at time $t = -\epsilon$, Alice and Bob verify that the state of the spins is $|0,0\rangle$. For the verification measurement, Alice has a device that interacts with her spin at a_V and Bob has a device that interacts with his spin at b_V. Since the verification is instantaneous, ϵ can be arbitrarily small. (Alice and Bob could even verify the state $|0,0\rangle$ many times before $t = 0$, but in Fig. 14.7 there is just one verification measurement.) So, without a doubt, the state of the spins up to $t = 0$ is $|0,0\rangle$; and without a doubt, after $t = 0$, the spins are in a product state. Thus, Alice and Bob directly verify that the collapse of $|0,0\rangle$ occurs on the $t = 0$ hyperplane, as the von Neumann collapse postulate assumes. But then collapse is not Lorentz invariant.

Thus observers in different frames disagree about collapse. Their disagreement, however, is not a contradiction. It would be a contradiction only if observers in different frames could verify incompatible accounts of collapse; but they cannot. Figure 14.8 illustrates this point. Alice and Bob verify the state $|0,0\rangle$ at a_V and b_V; in the frame of Figs. 14.7–8, a_V and b_V are simultaneous. Then at a, Alice measures the spin of her particle along some axis.[3] Now suppose two more observers, Alex and Barb, decide to verify the state $|0,0\rangle$ on the *same* pair of particles. Alex has a device that interacts with his particle (which is also Alice's particle) at a'_V, and Barb has a device that interacts with her particle (which is also Bob's particle) at b'_V. Then at b', Barb measures the spin of her particle along some axis. But Alex and Barb cannot verify the state $|0,0\rangle$ because the events a_V and b_V disturb the state of the spins. Likewise, Alice and Bob cannot verify the state $|0,0\rangle$ because, following the event a'_V, the spins are no

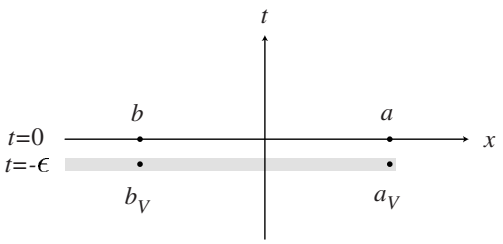

Figure 14.7: Alice and Bob verify the state $|0,0\rangle$ at time $t = -\epsilon$ and immediately afterwards, at $t = 0$, collapse the state with local measurements.

[3] How does the sequence of events look in another frame? In another frame, the verification of the entangled state $|0,0\rangle$ is not instantaneous. Since a_V and b_V are not simultaneous, the verification of $|0,0\rangle$ starts with one of the events a_V, b_V and continues until the other.

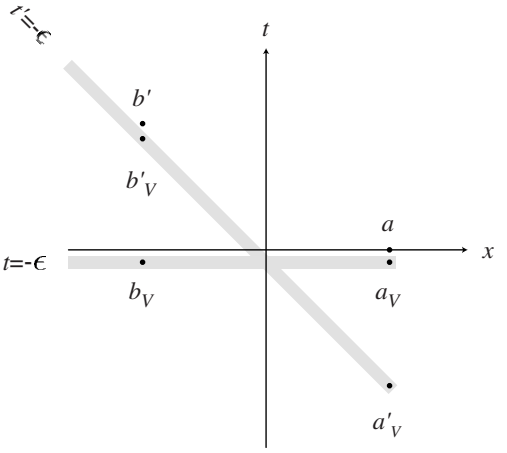

Figure 14.8: Alice and Bob try to verify the state $|0,0\rangle$ at time $t = -\epsilon$ while Alex and Barb try to verify the same state (on the same particles) at time $t' = -\epsilon$; a and b' are local measurements by Alice and Barb, respectively.

longer in this state. The two state-specific measurements disturb one another. If they did not disturb one another, Alice and Bob could verify that collapse occurs just after a_V and b_V, and Alex and Barb could verify that collapse occurs just after a'_V and b'_V. But they disturb each other, saving us from this contradiction.

Indeed, observers in different frames disagree not only about collapse. They also disagree about temporal order, length, energy, and all other physical quantities that are covariant but not invariant. The paradoxes of Sect. 14.1 prove only that collapse is not Lorentz invariant. In a Lorentz *covariant* collapse, observers in different frames could have different accounts of collapse. In any frame, each passing moment defines an equal-time hyperplane. If we reconsider the events in Figs. 14.4–6 we observe that, relative to any given frame, the singlet state $|\Psi_-\rangle$ always collapses on an equal-time hyperplane of that frame, i.e. on the equal-time hyperplane[4] that contains either a or b (whichever event is earliest). This observation is itself a covariant statement about collapse [8]. There is even a concrete model that corresponds to this statement of covariant collapse: it is the relativistic CSL model [9]. (See Sect. 9.2.) The classical field $w(\mathbf{x}, t)$ in the CSL model transforms as a Lorentz scalar (just as the probability of any measurement result is a Lorentz scalar), and the evolution equation for the state vector $|\psi_w(t)\rangle$ (a generalization of Eq. (9.10)) insures that collapse happens almost instantaneously, for a large enough measuring device, on an equal-time hyperplane in a given inertial reference frame. Covariant collapse resolves the paradoxes of Sect. 14.1.

All the same, relativity poses a problem for quantum measurements. We have seen that relativistic causality forbids an instantaneous measurement of almost all nonlocal operators and states. Hence the Hermitian operators, and the states, of relativistic quantum theory do not, in general, correspond to what experiments can measure. Ideally, a theory should predict what experiments can measure, neither more nor less. Relativistic constraints spoil the correspondence between quantum theory and what experiments can measure. This lack of correspondence may account for some of the pathology of relativistic quantum theory.

[4]More generally, we can state that collapse happens separately on every spacelike hypersurface containing the measurement event.

Problems

14.1 Show that nonlocal correlations of spin measurements on electrons, or polarization measurements on photons, are Lorentz invariant.

14.2 Show how Alice and Bob can modify the state-specific measurement of Sect. 14.3 to verify that their spin-1/2 particles are in the state $|2,0\rangle$.

*14.3 (a) Consider three observers – Alice, Bob and Claire – who share a triplet of spins in an entangled state. Suppose this state is the GHZ state:

$$|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle\right)$$

(See Sect. 3.4 and Prob. 3.13.) Show how Alice, Bob and Claire can, by local impulsive measurements, verify the GHZ state without changing it.
(b) Suppose that the triplet of spins is in the state

$$\alpha|\uparrow\uparrow\uparrow\rangle + \beta|\downarrow\downarrow\downarrow\rangle$$

with $|\alpha| \neq |\beta|$. Show that no instantaneous measurement can verify the state without changing it.

*14.4 (a) Show that any square matrix with complex entries is the product of a unitary matrix and a Hermitian matrix.
(b) Suppose that Alice and Bob share a pair of systems in an arbitrary state $|S\rangle$,

$$|S\rangle = \sum_{i,j=1}^{N} S_{ij}|a_i\rangle_A \otimes |b_j\rangle_B \ ,$$

where the states $|a_i\rangle_A$ are an orthonormal basis for Alice's system, the states $|b_j\rangle_B$ are an orthonormal basis for Bob's system, and the Hilbert space of each system has dimension N. Show that there are unitary transformations U_A on Alice's basis and U_B on Bob's basis such that

$$|S\rangle = \sum_{i=1}^{N} S^D_{ii}|a'_i\rangle_A \otimes |b'_i\rangle_B \ , \tag{14.14}$$

where $|a_i\rangle_A = U_A|a'_i\rangle_A$ and $|b_i\rangle_B = U_B|b'_i\rangle_B$ for every i, and where $S^D \equiv U_A S (U^*_B)^\dagger$ is diagonal with non-negative real entries. Equation (14.14) is known as the *Schmidt* or *polar* decomposition of $|S\rangle$.
(c) So far, Alice and Bob have done nothing to their systems; Eq. (14.4) is a mere rewriting of $|S\rangle$. Now let $|S\rangle$ be an entangled state of two identical systems having eigenstates $|1\rangle, |2\rangle, \ldots, |N\rangle$:

$$|S\rangle = \sum_{i,j=1}^{N} S_{ij}|i\rangle_A \otimes |j\rangle_B \ .$$

Show that if Alice and Bob apply local unitary transformations U_A and U_B, respectively, to their systems, they transform the state $|S\rangle$ to the diagonal form

$$U_A U_B |S\rangle = \sum_{i=1}^{N} S_{ii}^D |i\rangle_A \otimes |i\rangle_B \ .$$

14.5 Show that the conclusion of Sect. 14.4 is unchanged if the eigenstates of W that are orthogonal to $|\Psi_{\alpha\beta}\rangle$ are degenerate.

14.6 The measurement of $[S_z^{(A)} + S_z^{(B)}] \bmod 2\hbar$ in Sect. 14.5 assumes an entangled initial state of the measuring devices. The initial state must be an eigenstate of $Q_z^A + Q_z^B \bmod 2\hbar$ with eigenvalue 0 and an eigenstate of $e^{i(P_z^A \pm P_z^B)}$ with eigenvalue 1. Show that these requirements do not define the initial state uniquely.

14.7 Consider an operator with eigenstates

$$\begin{aligned} &|\uparrow\rangle_A \otimes |\uparrow\rangle_B \ , \\ &|\uparrow\rangle_A \otimes |\downarrow\rangle_B \ , \\ &|\downarrow\rangle_A \otimes |\uparrow'\rangle_B \ , \\ &|\downarrow\rangle_A \otimes |\downarrow'\rangle_B] \ , \end{aligned}$$

where $|\uparrow\rangle_A$ and $|\downarrow\rangle_A$ are spin states of Alice's particle and $|\uparrow\rangle_B$, $|\downarrow\rangle_B$, $|\uparrow'\rangle_B$ and $|\downarrow'\rangle_B$ are spin states of Bob's particle (with ${}_B\langle\uparrow' | \downarrow'\rangle_B = 0$). Show that if Alice and Bob could instantaneously measure this operator, then Alice could send a superluminal message to Bob, unless ${}_B\langle\uparrow | \uparrow'\rangle_B = 0$ or ${}_B\langle\uparrow | \downarrow'\rangle_B = 0$.

14.8 Consider an operator with four maximally entangled eigenstates, linear combinations of $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$, in the spin space of Alice's and Bob's particles. Show that Alice and Bob can, via local unitary operations, transform these eigenstates into the eigenstates of the operator W of Sect. 14.5.
(a) Prob. 14.4 implies that they can bring one of the eigenstates to the singlet form $|0,0\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. Show that any maximally entangled state orthogonal to $|0,0\rangle$ can be written (up to an overall phase)

$$\cos\theta \left[e^{i\varphi}|\uparrow\uparrow\rangle + e^{-i\varphi}|\downarrow\downarrow\rangle\right]/\sqrt{2} + i\sin\theta\,[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]/\sqrt{2} \ .$$

Show that either Alice or Bob can eliminate the phases $e^{\pm i\varphi}$ by applying a local unitary transformation, $e^{-i2\varphi S_z^{(A)}/\hbar}$ or $e^{-i2\varphi S_z^{(B)}/\hbar}$ respectively, that leaves $|0,0\rangle$ unchanged.
(b) Show that Alice and Bob, by applying $e^{-i\eta[S_x^{(A)} + S_x^{(B)}]/\hbar}$ (which leaves $|0,0\rangle$ unchanged), can transform the state

$$\cos\theta[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle]/\sqrt{2} + i\sin\theta[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]/\sqrt{2}$$

into the triplet state $|2,0\rangle$ if $\eta = -\theta$ or the state $[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle]/\sqrt{2}$ if $\eta = \pi/2 - \theta$. (By orthogonality, the last eigenstate must be $[|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle]/\sqrt{2}$.)

14.9 (a) Consider an "exchange" measurement which instantaneously (and locally) exchanges the spin states of two particles, one of which is treated as a measuring device. For example, Alice could prepare a "measuring" spin in an initial state $|\uparrow\rangle_d$ and her own particle's spin in an initial state $|\psi\rangle_A$; an exchange measurement would transform the initial state $|\psi\rangle_A \otimes |\uparrow\rangle_d$ into $|\uparrow\rangle_A \otimes |\psi\rangle_d$. Show that the transformation is unitary. (b) Consider a second such "exchange" measurement, which Bob could apply to the spin state of his particle. Together, the two exchange measurements would instantaneously transfer an arbitrary spin state $|\Psi_{\alpha\beta}\rangle$ of Alice's and Bob's particles to the "measuring" particles. Then the "measuring particles", when reunited, would allow a verification of the state $|\Psi_{\alpha\beta}\rangle$. Would this state-specific (state-verification) measurement violate relativistic causality? Explain.

14.10 Show that the two state-verification procedures in Fig. 14.8 disturb one another, so that neither verifies the singlet state $|0,0\rangle$. Why does the event a not disturb the state verification of Alex and Barb?

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15 How to Observe a Quantum Wave

In 1926, Schrödinger postulated "material waves" in an analogy with light waves. Rays of light – which obey the principle of least time – are an approximation to light waves. In Schrödinger's analogy, paths of material objects – which obey the principle of least action – are an approximation to material waves. The paths of material objects are as fictitious as rays of light, while material waves are as real (and as measurable) as light waves [1]. But within months, Born had discarded "the physical pictures of Schrödinger, which aim at a revitalization of the classical continuum theory" [2], and given the "material wave" $\Psi(\mathbf{x}, t)$ a new interpretation: $|\Psi(\mathbf{x}, t)|^2$ is the probability density to find a particle at $\mathbf{x}$ at time t. Born's interpretation suggests that for a *single* object, $\Psi(\mathbf{x}, t)$ is *not* measurable – to measure a probability density, we need to prepare $\Psi(\mathbf{x}, t)$ many times.

We claim, throughout this book, that nonrelativistic quantum mechanics corresponds completely with experiment (whenever we can treat the speed of light as infinite). If so, what corresponds to the quantum wave? That is, what corresponds to "the wave function of a single electron" if no experiment on a single electron can measure $\Psi(\mathbf{x}, t)$ (or even $|\Psi(\mathbf{x}, t)|^2$)? Our answer to this question would have surprised both Schrödinger and Born: experiments *can* measure the wave function of a single electron! This chapter explains how (and in what sense) "protective" measurements on a single electron yield both the norm and the relative phase of $\Psi(\mathbf{x}, t)$.

Protective measurements belong to, and extend, nonrelativistic quantum mechanics. Are they consistent with relativistic causality? Two paradoxes in Sect. 15.1 hint that, in general, quantum measurements are *not* consistent with relativistic causality. And in particular, how can a measurement of $\Psi(\mathbf{x}, t)$ – an extended object that collapses instantaneously – be consistent with relativistic causality? (See Sect. 14.1.)

15.1 Dipole Paradox

Einstein, in his debate with Bohr, used a version of the two-slit interference experiment to argue that quantum mechanics is inconsistent. Bohr pointed out a flaw in Einstein's argument. (See Sect. 2.4.) A more sophisticated version of Einstein's argument, in Sect. 4.1, is flawed, as well; but let us now try a still more sophisticated version. Figure 15.1 shows the experiment. Electrons in an eigenstate of momentum pass through a screen with two slits and strike a row of detectors, which record an interference pattern. There is at most one electron in the apparatus at a time – the time of flight t_0 of an electron through the apparatus is much less than the time T between successive clicks of the detectors. Standing a distance $L > ct_0/2$ from the apparatus

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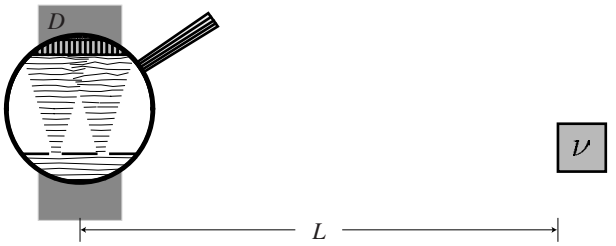

Figure 15.1: An apparatus, located at D, in which electrons pass one at a time through a two-slit interference experiment; and Prof. Nu's device, located at ν, for measuring electric field strength.

is a physicist, Prof. Nu, with a device for measuring electric field strength. Nu stands closer to the right slit than to the left slit. Thus her measuring device detects a stronger electric field if the electron passes through the right slit than if it passes through the left slit, and she can check through which slit the electron passes.

In the other versions of the two-slit interference experiment, attempts to check which way the electrons go spoil the interference pattern. But Nu's measurements of the electric field *cannot* spoil the interference pattern. Consider one such measurement: At time $t = 0$, an electron reaches the two slits. To find out through which slit the electron passes, Nu measures the electric field at time $t = L/c$; she must wait that long for the field of the electron passing through the apparatus to reach her. Does Nu's measurement of the electric field affect the electron? Whatever Nu does at time $t = L/c$ cannot affect the electron before the time $t = 2L/c$, according to relativistic causality. But by the time $t = 2L/c > t_0$, the electron has already reached the row of detectors and made one of them click, so Nu's measurement cannot possibly affect the detection of the electron.

We have treated Nu's measurement as a single spacetime event. Can we? The answer depends on her measuring device. If her measuring device is a large, charged sphere, as in Sect. 8.4, then we cannot: the field of the sphere might affect the two-slit apparatus long before the electron passes through it. But suppose her measuring device is a collapsible dipole – two equal and opposite charges that separate to form a dipole and then rejoin. At time $t = L/c$, Nu opens the dipole, measures the electric field via its effect on each charge, and closes the dipole. Except for a short time around $t = L/c$, the measuring device is electrically neutral and cannot affect the electron passing through the apparatus.

At time $t = T$, another electron passes through the apparatus. We can choose T large enough such that any fields due to Nu's first measurement have dissipated. Once again, Nu checks through which slit the electron passes, with no effect on the electron before its detection. A third electron passes through at $t = 2T$, a fourth at $t = 3T$, and so on. After many electrons, the detectors show a diffraction pattern. Yet Nu can tell us through which slit each electron went, can't she? Nu? Relativistic causality allows Nu to violate Bohr's complementarity principle.

Here is a related paradox, in which Prof. Nu violates Heisenberg's uncertainty relations. At time $t = 0$, a particle with charge Q is located somewhere inside Prof. Nu's large laboratory. Her students have just made very accurate measurements of the particle's momentum, but all they know about its position is that the particle is still somewhere in the laboratory. Nu herself

is standing a distance L from the laboratory, where she measures the local electric field by opening a dipole for a short time around $t = L/c$. From this measurement she infers the particle's position at $t = 0$. At time $t = L/c$ she also receives a report from her students regarding the particle's momentum at $t = 0$. Nu's position measurement can have no effect on the particle until time $t = 2L/c$. Thus, during the time interval $L/c < t < 2L/c$, Nu knows the position and momentum of the particle more accurately than the uncertainty relations allow.

How do we resolve these paradoxes? We may wonder whether we have treated the electric field correctly. Is it a classical or a quantum field? We have to ask, because it is inconsistent to couple a quantum particle to a classical electromagnetic field. An electron passing through the apparatus generates an electric field. If this electric field is classical, it cannot be in a quantum superposition. But the electron passing the two slits *is* in a quantum superposition – it has a nonzero amplitude to pass through each slit. If the electric field coupled to the electron were classical, it would instantly collapse the quantum superposition of the electron, and there could be no interference pattern. So the electric field, too, must be in a quantum superposition. It must be a quantum field. (See Sect. 8.4.) This argument applies to any field coupled to the electron, including the gravitational field [3]. But does a *quantum* electric field help us resolve the paradoxes?

15.2 How *not* to Observe a Quantum Wave

Born's probability interpretation of $\Psi(\mathbf{x}, t)$ suggests that $\Psi(\mathbf{x}, t)$ is measurable only if we prepare it many times. We do not *know* that $\Psi(\mathbf{x}, t)$ is measurable only if we prepare it many times; we only know that if we *do* prepare it many times, we can measure $|\Psi(\mathbf{x}, t)|^2$. However, there are at least four good arguments why $\Psi(\mathbf{x}, t)$ is not measurable if we prepare it only once. Let us review them.

The first argument comes from experience. Born's interpretation fits what experiments show us. Cathode rays leave tracks of tiny water droplets as they cross a Wilson cloud chamber – straight tracks or (in the presence of magnetic fields) curved tracks. As Schrödinger [4] himself put it many years later, "We cannot but interpret them as traces of the paths of single electrons". And although the quantum wave of an electron spreads out in space, we never see the electron itself spread out in space; we never detect it in two or more places simultaneously.

Unitarity provides a second argument. Imagine a device that couples to a single electron wave function $\Psi(\mathbf{x}, t)$ and, let us assume, measures $|\Psi(\mathbf{x}, t)|^2$. The device need not measure $|\Psi(\mathbf{x}, t)|^2$ all over spacetime; let us assume only that it measures the wave function at $\mathbf{x} = \mathbf{0}$, $t = 0$. Suppose the device has a pointer that, at the beginning of the experiment, points to the value 0 on a dial; at the end of the measurement, it points to the value $|\Psi(\mathbf{0}, 0)|^2$. If the wave function of the electron is $\Psi_1(\mathbf{x}, t)$, the pointer points to $|\Psi_1(\mathbf{0}, 0)|^2$, while if the wave function of the electron is $\Psi_2(\mathbf{x}, t)$, the pointer points to $|\Psi_2(\mathbf{0}, 0)|^2$. Now, these two states of the pointer are orthogonal, but the wave functions $\Psi_1(\mathbf{x}, t)$ and $\Psi_2(\mathbf{x}, t)$ need not be orthogonal; and if they are not orthogonal, the device violates unitarity – it transforms nonorthogonal states into orthogonal states.

Indeed, linearity alone provides an argument. Suppose the device interacts with an electron in a superposition $[\Psi_1(\mathbf{x}, t) + \Psi_2(\mathbf{x}, t)]/\sqrt{2}$. Linear time evolution would evolve the pointer to a superposition of pointing to $|\Psi_1(\mathbf{0}, 0)|^2$ and to $|\Psi_2(\mathbf{0}, 0)|^2$. But if the quantum wave is

measurable, the pointer should *not* be in a superposition at the end of the measurement; it should point to the value

$$\frac{1}{2}|\Psi_1(\mathbf{0},0)+\Psi_2(\mathbf{0},0)|^2 \ ,$$

which is the probability density of the superposition $[\Psi_1(\mathbf{x},t)+\Psi_2(\mathbf{x},t)]/\sqrt{2}$ at $\mathbf{x}=\mathbf{0}$, $t=0$. Thus, by assuming that an arbitrary quantum wave is measurable, we contradict the linearity of quantum evolution.

The fourth argument comes from relativistic causality (and the paradoxes of Sect. 14.1). Consider a quantum wave that extends over a macroscopic domain. For example, let our friends Alice and Bob each have a box, and let them come together to prepare an electron in a quantum superposition $[\Psi_A(\mathbf{x},t)+\Psi_B(\mathbf{x},t)]/\sqrt{2}$; here $\Psi_A(\mathbf{x},t)$ is a normalized state with support only in Alice's box, and $\Psi_B(\mathbf{x},t)$ is a normalized state with support only in Bob's box. Alice and Bob then go off in opposite directions, with their boxes, to a very great distance. Alice also takes with her a device for measuring the quantum wave, while Bob takes with him a Geiger counter. At time $t=0$, Alice (at $\mathbf{x}=\mathbf{0}$) measures the quantum wave (or the corresponding probability density) inside her box. Just *before* $t=0$ (in some reference frame) Bob may switch on the Geiger counter and look for the electron in his box, or he may do nothing. If he does nothing, Alice's measurement will yield the value $|\Psi_A(\mathbf{0},0)|^2/2$. But if Bob chooses to switch on the counter, he may find the electron in his box, and then Alice's measurement will yield a zero result; or he may not find the electron in his box, and then Alice's measurement will yield the result $|\Psi_A(\mathbf{0},0)|^2$ (assuming perfect efficiency for the Geiger counter). Either way, Bob can send a message to Alice – and violate relativistic causality – by choosing whether or not to look for the electron.

With such weighty arguments – unitarity, linearity, relativistic causality and decades of experience – to show that $\Psi(\mathbf{x},t)$ is *not* measurable, it would seem a waste of time to try to show that it *is*. But that is what we do next. Only afterward – after showing that $\Psi(\mathbf{x},t)$ is measurable – do we look for loopholes in these four arguments.

15.3 Protective Measurements

Section 7.2 sets out the paradigm for a quantum measurement. We postulate a measurement interaction

$$H_{int}(t)=g(t)A_sP_d \ ,$$

in which A_s is the measured observable; P_d is canonically conjugate to an observable Q_d representing the pointer position on the measuring device. The coupling $g(t)$ is different from zero only for times $0\le t\le T$ and normalized according to

$$\int_0^T g(t)\,dt=g_0 \ .$$

Thus, the whole measurement lasts no longer than T. Often, it is convenient to take the impulsive limit $T\to 0$. But let us consider the opposite limit, in which T is very large, and $g(t)$ is always near 0, changing slowly. What does the measurement yield in this limit?

Suppose that H_s (the Hamiltonian of the system when it is not measured) has discrete and nondegenerate energy eigenvalues E_i, and the system is in one of the corresponding energy eigenstates $|\Psi_i\rangle$. If $|\Psi_i\rangle$ is in an eigenstate of A_s, the measurement (however long it takes) yields the corresponding eigenvalue of A_s. If $|\Psi_i\rangle$ is not an eigenstate of A_s, let us treat $H_{int}(t)$ as a perturbation $g_0 A_s P_d/T$, lasting a time T, on H_s. To first order [5] in $1/T$, the state of the system remains $|\Psi_i\rangle$; the corresponding eigenvalue of $H_s + H_{int}$ is $E_i + \langle\Psi_i|H_{int}|\Psi_i\rangle$. The shift in the energy is $\langle\Psi_i|H_{int}|\Psi_i\rangle$ and equals $g_0 P_d/T$ times the expectation value $\langle\Psi_i|A_s|\Psi_i\rangle$. So the pointer on the measuring device shifts by $g_0\langle\Psi_i|A_s|\Psi_i\rangle$ during the measurement – a measurement on a *single* system can yield an expectation value!

Can it? Let us apply the adiabatic theorem to the measurement. The combination $H_s + H_{int}(t)$ is the slowly changing Hamiltonian of the system; it depends on time only through the small coupling $g(t)$. Applying the formalism of Sect. 12.2, we obtain an effective Hamiltonian H_{eff} for the pointer of the measuring device:

$$H_{eff} = H_s + g(t) P_d \sum_i \mathbf{\Pi}_i A_s \mathbf{\Pi}_i \ ,$$

where $\mathbf{\Pi}_i = |\Psi_i\rangle\langle\Psi_i|$ projects onto $|\Psi_i\rangle$. (For simplicity, we take the Hamiltonian H_d of the measuring device to vanish when there is no measurement.) The equation of motion for Q_d is

$$\frac{d}{dt} Q_d = g(t) \sum_i \mathbf{\Pi}_i A_s \mathbf{\Pi}_i \ ,$$

which we integrate to obtain

$$Q_d(T) - Q_d(0) = g_0 \sum_i \mathbf{\Pi}_i A_s \mathbf{\Pi}_i \ .$$

If the system is in the i-th energy eigenstate, the change in the pointer reading is g_0 times the expectation value $\langle\Psi_i|A_s|\Psi_i\rangle$. Thus we can measure the expectation value of A_s on a single system.

We call such a measurement [6] *protective* because it protects the quantum wave $\Psi_i(\mathbf{x})$ from the measurement. Non-adiabatic measurements of A_s cause the measured system to jump to an eigenstate of A_s; but here, in the adiabatic limit, the system does not jump. As long as the measurement interaction is weak enough, the system remains in the same energy eigenstate.

Before, we knew how to measure an expectation value $\langle\Psi|A_s|\Psi\rangle$ only by preparing the quantum state $|\Psi\rangle$ many times and measuring A_s each time. We assumed that we could not prepare $|\Psi\rangle$ once and measure $\langle\Psi|A_s|\Psi\rangle$, just as we assumed that we cannot prepare $|\Psi_i\rangle$ once and measure $|\Psi_i(0)|^2$. But if we can prepare $|\Psi\rangle$ once and measure $\langle\Psi|A_s|\Psi\rangle$, perhaps we can also measure $|\Psi_i(0)|^2$, for $|\Psi_i(0)|^2$ is itself an expectation value – the expectation value of $\delta(\mathbf{x})$. That is, we can measure the average value of $|\Psi_i(\mathbf{x})|^2$ in an arbitrary neighborhood as an expectation value. For example, we can measure the value of $|\Psi_i(\mathbf{x}_0)|^2$ as the expectation value of $\rho_D(\mathbf{x} - \mathbf{x}_0)$, where $\rho_D(\mathbf{x} - \mathbf{x}_0)$ is

$$\rho_D(\mathbf{x} - \mathbf{x}_0) = \frac{e^{-|\mathbf{x}-\mathbf{x}_0|^2/D^2}}{\pi^{3/2} D^3}$$

and D is arbitrary. The distribution $\rho_D(\mathbf{x}-\mathbf{x}_0)$ approaches $\delta(\mathbf{x}-\mathbf{x}_0)$ as $D \to 0$. (As $D \to 0$, the interaction $H_{int}(t)$ grows stronger, so the duration T of the measurement must also grow.)

We can generalize the measurement of $\Psi_i(\mathbf{x})$ in several ways. We are not limited to measuring $\Psi_i(\mathbf{x})$ in the neighborhood of one point. A measurement interaction with several terms,

$$g(t)\sum_i \rho_D(\mathbf{x}-\mathbf{x}_i)P_d^{(i)} \ ,$$

allows us to measure the value of $|\Psi_i(\mathbf{x})|^2$ at several points $\mathbf{x}_i$ simultaneously. Again, the duration of the measurement must grow but, given enough time, we can map the quantum wave as completely and as precisely as we wish.[1] We can also measure the average current density in a neighborhood of a point $\mathbf{x}_0$ by a protective measurement of an operator $\mathbf{J}_D(\mathbf{x}_0)$:

$$\mathbf{J}_D(\mathbf{x}_0) = \frac{\hbar}{2im}\left[\rho_D(\mathbf{x}-\mathbf{x}_0)\nabla + \nabla\rho_D(\mathbf{x}-\mathbf{x}_0)\right] \ . \tag{15.1}$$

A measurement of this operator yields the probability current $\mathbf{J}_i(\mathbf{x})$,

$$\mathbf{J}_i(\mathbf{x}) = \frac{\hbar}{2im}\left[\Psi_i^*(\mathbf{x})\nabla\Psi_i(\mathbf{x}) - \Psi_i(\mathbf{x})\nabla\Psi_i^*(\mathbf{x})\right] \ , \tag{15.2}$$

averaged over the distribution $\rho_D(\mathbf{x}-\mathbf{x}_0)$. In the notation of Sect. 4.2, a quantum wave comprises a modulus $n(\mathbf{x},t)$ and a phase $\varphi(\mathbf{x},t)$,

$$\Psi(\mathbf{x},t) = n(\mathbf{x},t)e^{i\varphi(\mathbf{x},t)} \ ;$$

a stationary state depends on t only through a global phase, so $n = |\Psi_i(\mathbf{x})|$ and $\nabla\varphi = m\mathbf{J}_i(\mathbf{x})/|\Psi_i(\mathbf{x})|^2$. Thus by measuring the expectation values of $|\Psi_i(\mathbf{x})|^2$ and $\mathbf{J}_i(\mathbf{x})$, we can obtain $\nabla\varphi$ and reconstruct the relative phase φ as precisely as we wish.

A Hamiltonian H_s with a discrete, nondegenerate spectrum naturally protects its eigenstates during a measurement; but even without such a Hamiltonian, we can protect an arbitrary quantum wave $\Psi(\mathbf{x},t)$. Chapter 12 shows that both the adiabatic limit and the limit of continuous measurement suppress quantum jumps. Thus, we can protect $\Psi(\mathbf{x},t)$ by continuous (i.e. dense) measurements of any operator that, at time t, has $|\Psi(t)\rangle$ as a nondegenerate eigenstate with an isolated eigenvalue, e.g. the projection operator $|\Psi(t)\rangle\langle\Psi(t)|$. With continuous measurement we can obtain an arbitrary evolution of the quantum wave, including the evolution that $\Psi(\mathbf{x},t)$ would have had in the absence of any measurement.

15.4 Galilean Dialogue

[Salviati, Sagredo and Simplicio are visiting Ma'alot-Tarshiha.]

SAGREDO. These protective measurements are a great revelation to me; yet, when I try to grasp the revelation, it slips through my fingers like sand. I want to perform a protective

[1] Instead of increasing the duration of the measurement, we can increase the gap between the energy of $|\Psi_i\rangle$ and other eigenvalues of H_s. Indeed, the larger we make the gap, the more we can speed up a protective measurement; in principle, there is no maximum size for the gap, hence no minimum time for a protective measurement.

measurement on an unknown quantum wave $\Psi_i(\mathbf{x})$. But in fact, I cannot perform this protective measurement, because I cannot protect $\Psi_i(\mathbf{x})$. To protect $\Psi_i(\mathbf{x})$ I must know that $\Psi_i(\mathbf{x})$ is an eigenstate of a Hamiltonian H_s, that the corresponding eigenvalue is nondegenerate, and that a sufficiently large gap separates it from other eigenvalues of H_s. Then I can protect $\Psi_i(\mathbf{x})$ with an adiabatic measurement. But if I know all these things, I can just as well measure the observable H_s and identify $\Psi_i(\mathbf{x})$ from its eigenvalue. I could then calculate the modulus and phase of $\Psi_i(\mathbf{x})$, and even calculate $\langle\Psi_i|A_s|\Psi_i\rangle$ for any operator A_s. Why do we gain from protective measurements?

SALVIATI. I appreciate your question, Sagredo; it's a question that I asked myself, as well. Let me point out that you *can*, in a sense, perform a protective measurement on an unknown quantum wave. For example, you could leave the protection to me. If *I* know the Hamiltonian H_s, I can protect $\Psi_i(\mathbf{x})$. You can then measure the modulus and phase of $\Psi_i(\mathbf{x})$ without even knowing that $\Psi_i(\mathbf{x})$ is an energy eigenstate. I only have to tell you how adiabatic to make your measurement. Think about this example, and you will agree with me that protective measurements are unlike any familiar measurements.

SAGR. I don't doubt that protective measurements are unlike any familiar measurements. But I doubt that we need them. In your example, you and I play a measurement game. But to me, measurement is a game that we physicists play with Nature. What do protective measurements tell us about Nature?

SALV. Well, I see your point. And I can answer your question: Protective measurements allow us to test, for the first time, whether the quantum wave $\Psi_i(\mathbf{x})$ and the "expectation value" $\langle\Psi_i|A_s|\Psi_i\rangle$ describe a *single* system in the state $|\Psi_i\rangle$, or only an ensemble of systems in the state $|\Psi_i\rangle$. Consider a single particle in a square well trapped in, say, the ground state $|\Psi_0\rangle$. If we look for the particle in the left half of the box, we either find it or we don't. We never obtain 1/2, which is the expectation value of the operator $\mathbf{\Pi}_L$ that projects onto the left half of the box. Only by measuring $\mathbf{\Pi}_L$ many times on the state $|\Psi_0\rangle$ and averaging do we approach 1/2. By contrast, a *single* protective measurement of $\mathbf{\Pi}_L$ on a particle prepared *once* in the state $|\Psi_0\rangle$ yields $\langle\Psi_0|\mathbf{\Pi}_L|\Psi_0\rangle = 1/2$.

Nature protects, too. Consider a system in its ground state $|\Psi_0\rangle$. If the ground state energy is nondegenerate and discrete, then $|\Psi_0\rangle$ is naturally protected. Anything in the system's environment that couples to an observable A_s of the system measures, in effect, $\langle\Psi_0|A_s|\Psi_0\rangle$.

SAGR. Your answer convinces me.

SIMPLICIO. But not me! Salviati, you say that by measuring $\mathbf{\Pi}_L$ many times on the state $|\Psi_0\rangle$ and averaging, we obtain the expectation value $\langle\Psi_0|\mathbf{\Pi}_L|\Psi_0\rangle$. My intuition is hopelessly classical, but isn't a protective measurement of $\mathbf{\Pi}_L$ equivalent to a repeated measurement of $\mathbf{\Pi}_L$ on the state $|\Psi_0\rangle$? The adiabatic limit keeps the particle in the ground state, while the device measuring $\mathbf{\Pi}_L$ obtains 0 at some times, 1 at other times; what it ultimately, and automatically, registers is the time average of $\mathbf{\Pi}_L$, i.e. the expectation value of $\mathbf{\Pi}_L$.

SALV. In quantum mechanics, a measurement of $\mathbf{\Pi}_L$ cannot yield 0 or 1 and still leave the particle in the state $|\Psi_0\rangle$.

Now, you could assume a theory of hidden variables But let me show you how strange such a theory would be. Let us prepare the particle, not in $|\Psi_0\rangle$, but in the first excited state $|\Psi_1\rangle$. As you know, $|\Psi_1\rangle$ has a node in the middle of the box, and the probability density vanishes there. Since the probability density does not vanish on either side of the node, the particle must spend time on both sides of the node; in fact, it must always be crossing the node to spend

time on the other side. However, the probability density vanishes at the node, so the particle would have to travel with infinite speed at the node, in order to cross from one side of the box to the other without spending time there. If the particle is an electron, it would radiate as it accelerated near the node. Yet the potential is flat; no force acts on the particle.

SAGR. That is strange, I agree.

SIMP. And so do I; but I would still prefer a measurement allowing me to observe a completely unknown quantum wave. Indeed, using your method I think I can design such a measurement. I can probe the energy levels of an unknown system spectroscopically. Once I know its energy levels, I can design a protective measurement on it, even if I do not know its Hamiltonian. And if the system is a particle with a Hamiltonian of the form $H_s = p^2/2m + V(\mathbf{x})$, I can reconstruct $V(\mathbf{x})$ from protective measurements of its charge distribution.

SALV. Such a measurement is indeed possible, Simplicio. But it would not allow us to observe every unknown quantum wave. Alas, unitarity does not allow us to observe an arbitrary unknown quantum wave.

15.5 Protective Measurements and Causality

Now we have some explaining to do. How do protective measurements get around the four arguments of Sect. 15.2? What is wrong with the arguments?

The argument from experience merely shows the limitations of our experience. We must broaden our experience! By protecting the state of the measured system, protective measurements succeed where other measurements fail.

The second argument – from unitarity – is correct as far as it goes. (Salviati refers to it at the end of the last section.) But it does not prevent us from distinguishing a state $|\Psi_i\rangle$ from states *orthogonal* to $|\Psi_i\rangle$. Any form of protection selects a set of orthogonal states. A nondegenerate Hamiltonian selects its own (orthogonal) eigenstates; continuous observation of a nondegenerate observable selects the (orthogonal) eigenstates of the observable. If the selected states are $|\Psi_n\rangle$, a protective measurement *must* find the system in one of the states $|\Psi_n\rangle$, hence it only distinguishes the state of the system from orthogonal states.

But if we can protect any one of the states $|\Psi_n\rangle$, why can't we protect a superposition of the states $|\Psi_n\rangle$? Let us prepare a system in a superposition $|\Psi_{\alpha\beta}\rangle$ of orthogonal states:

$$|\Psi_{\alpha\beta}\rangle = \alpha|\Psi_1\rangle + \beta|\Psi_2\rangle \ ,$$

with $|\alpha|^2 + |\beta|^2 = 1$. Why can't we protect $|\Psi_{\alpha\beta}\rangle$? A measurement that protects $|\Psi_1\rangle$ and $|\Psi_2\rangle$ cannot change $|\alpha|$ and $|\beta|$ (because it suppresses quantum jumps). However, a protective measurement on the system leaves the measuring device entangled with the measured system. Let $\Phi(Q_d, 0)$ denote the initial wave function of the measuring device, and let $\Phi(Q_d, 0)$ be peaked at $Q_d = 0$. The combined initial wave function is $\Psi_{\alpha\beta}(\mathbf{x}, 0)\Phi(Q_d, 0)$, which evolves to

$$\begin{aligned}\alpha\Psi_1(\mathbf{x}, T)\Phi\left(Q_d - g_0\langle\Psi_1|A_s|\Psi_1\rangle, T\right) \\ + \beta\Psi_2(\mathbf{x}, T)\Phi\left(Q_d - g_0\langle\Psi_2|A_s|\Psi_2\rangle, T\right)\end{aligned} \tag{15.3}$$

over time T. Each of the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ is protected, but there is nothing that protects the entangled state of the measured system and the measuring device, and it collapses. The measur-

ing device does not remain in a macroscopic superposition but points to $Q_d = g_0\langle\Psi_1|A_s|\Psi_1\rangle$ with probability $|\alpha|^2$, or to $Q_d = g_0\langle\Psi_2|A_s|\Psi_2\rangle$ with probability $|\beta|^2$.

The third argument – from linearity – is correct as far as it goes, too. The evolution of $\Psi_{\alpha\beta}(\mathbf{x},0)\Phi(Q_d,0)$ is indeed linear, yielding the superposition in Eq. (15.3). Thus the measuring device points either to $g_0\langle\Psi_1|A_s|\Psi_1\rangle$ or to $g_0\langle\Psi_2|A_s|\Psi_2\rangle$ at the end of the measurement. It does not point to $g_0\langle\Psi_{\alpha\beta}|A_s|\Psi_{\alpha\beta}\rangle$ and we do not expect it to point to $g_0\langle\Psi_{\alpha\beta}|A_s|\Psi_{\alpha\beta}\rangle$; the quantum wave $\Psi_{\alpha\beta}(\mathbf{x},t)$ is not protected, so we cannot measure it.

The fourth argument invokes relativistic causality. If Alice and Bob can protect the state $[|\Psi_A\rangle + |\Psi_B\rangle]/\sqrt{2}$, then Bob can instantly change the result of Alice's measurement. However, the normalized wave functions $\Psi_A(\mathbf{x},t)$ and $\Psi_B(\mathbf{x},t)$ are disjoint, by assumption. If the Hamiltonian H_s of their shared electron is local, then $\langle\Psi_A|H_s|\Psi_B\rangle$ vanishes. It follows that $[|\Psi_A\rangle + |\Psi_B\rangle]/\sqrt{2}$ is degenerate with $[|\Psi_A\rangle - |\Psi_B\rangle]/\sqrt{2}$, and the Hamiltonian of the electron does not protect the state $[|\Psi_A\rangle + |\Psi_B\rangle]/\sqrt{2}$. Alice and Bob must protect their state some other way. But the other ways involve explicitly nonlocal interactions. For example, Alice and Bob can remove the degeneracy between $[|\Psi_A\rangle + |\Psi_B\rangle]/\sqrt{2}$ and $[|\Psi_A\rangle - |\Psi_B\rangle]/\sqrt{2}$ by introducing an interaction between $|\Psi_A\rangle$ and $|\Psi_B\rangle$ in H_s. But such a nonlocal interaction violates relativistic causality.

Here is a related, and subtler, argument. For simplicity, we confine the electron to the x-axis. Alice's box is a one-dimensional potential well, and we assume that outside the box the potential vanishes, but the electron wave function does not. Alice prepares the electron in the ground state $|\Psi_0\rangle$ of the well, with energy $-E$. She measures $|\Psi_0(0)|^2$, the expectation value of $|\Psi_0(x)|^2$ in the center of the well, and finds a nonzero value. If she repeats the measurement, she finds the same value. Now suppose that Bob decides to look for the electron a distance L along the x-axis from the potential well. It is very unlikely that Bob will find the electron, for the probability density $|\Psi_0(L)|^2$ is exponentially small; we have

$$|\Psi_0(L)|^2 \approx \frac{1}{\hbar}(2mE)^{1/2}e^{-2L(2mE)^{1/2}/\hbar}\,, \tag{15.4}$$

where m is the mass of the electron. But $|\Psi_0(L)|^2$ is not zero, so Bob might find the electron. If he does, Alice will obtain zero for her measurement of $|\Psi_0(0)|^2$. Thus Bill can send a superluminal message to Alice. Bob has only a small chance of transmitting the message, but Alice and Bob can improve the chance of transmission by preparing an ensemble of electrons in the state $|\Psi_0\rangle$.

This thought experiment is a paradox if the duration T of Alice's protective measurement is smaller than the time it takes for a light signal to reach her from Bob, i.e. if we assume $T < L/c$. But we applied the Born–Oppenheimer approximation in our treatment of protective measurements; we have taken large T for granted. What does the upper bound $T < L/c$ do to protective measurements? If T is too short, the protective measurement may not be adiabatic; then Alice's measurement may kick the electron out of the well. That is, she may obtain $|\Psi_0(0)|^2 = 0$ not because Bob found the electron but because her measurement was not adiabatic, and excited the electron to a state with positive energy. So we ask, what is the minimum probability that Alice's measurement kicks the particle out of well, if the measurement lasts a time $T < L/c$? If this probability is larger than the probability of Bob's finding the particle a distance L from the well, we have no paradox; Alice cannot conclude

that Bob is sending her a message when her measurement of $|\Psi_0(0)|^2$ yields zero. It is more likely that the measurement itself kicked the particle out of the well.

What is the minimum probability that a measurement lasting a time T will cause a quantum jump in energy of at least E? Let us try to estimate the minimum probability. (See also Probs. 15.4–5.) It is sufficient to consider a system with two nondegenerate energy eigenstates; additional states could only increase the probability of a transition. Hence we assume a measurement interaction of the form

$$H_{int} = g(t) A_s P_d \ ,$$

where A_s is a 2-by-2 matrix. If $g(t)$ is an analytic function of t there are general methods to calculate the probability of a transition, and a few exact results [7]. Here $g(t)$ cannot be an analytic function, because an analytic function of t cannot vanish over any interval in t, and H_{int} is nonzero for a time interval not longer than T. However, if $g(t)$ is analytic and changes gradually enough, H_{int} is less likely to induce quantum jumps than if $g(t)$ has finite support. Suppose the coupling is

$$g(t) = \frac{g_0}{T} \mathrm{sech} \frac{\pi t}{T} \ , \tag{15.5}$$

which is exponentially small except during a period of time of order T, and normalized according to

$$\int_{-\infty}^{\infty} g(t) \, dt = g_0 \ .$$

(See Fig. 15.2.) Let $H_s = E\sigma_z/2$ be the unperturbed Hamiltonian of the model, with eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$, and let $H_{int} = g(t)\sigma_x P_d$. At time $t = -\infty$ the combined state of the measured system and the measuring device is $|\downarrow\rangle \otimes |\Phi_d\rangle = |\downarrow, \Phi_d\rangle$, where $\Phi_d(Q_d)$ is the wave function of the pointer position Q_d. The probability of a transition is

$$P_{jump} = |\langle\uparrow| e^{-i\int_{-\infty}^{\infty} dt[H_s + H_{int}(t)]/\hbar} |\downarrow, \Phi_d\rangle|^2 \ .$$

We apply the exact calculation of N. Rosen and C. Zener [8], who assumed the coupling of Eq. (15.5), to obtain

$$P_{jump} = \langle\Phi_d| \sin^2(g_0 P_d/\hbar) \mathrm{sech}^2(ET/2\hbar) |\Phi_d\rangle \ .$$

The state $|\Phi_d\rangle$ is not an eigenstate of P_d; but the exponential factor in P_{jump} does not depend on the expectation value of $\sin^2(g_0 P_d/\hbar)$; we have

$$P_{jump} \approx e^{-ET/\hbar} \tag{15.6}$$

as an estimate of the minimum probability of a transition due to Alice's protective measurement.[2]

[2] Although the probability of $|\uparrow\rangle$ in the final state is exponentially small, at intermediate times the coefficient of $|\uparrow\rangle$ in the state of the measured system may not be exponentially small. (See Prob. 15.8.) Yet at time T the coefficient is exponentially small and Alice's measuring device is not entangled with the measured system.

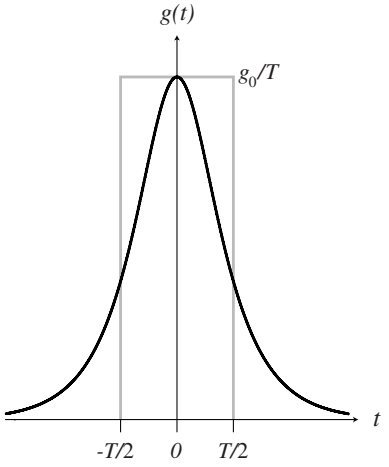

Figure 15.2: The coupling $g(t)$ of Eq. (15.5).

Now we compare the exponents in Eqs. (15.4) and (15.6). The probability that Alice's measurement kicks the electron out of the well, assuming that her measurement is as adiabatic as possible in the time T, is of order $e^{-ET/\hbar}$. For a paradox, this probability must be smaller than the probability that Bob finds the electron a distance L from the well, i.e.

$$e^{-ET/\hbar} < e^{-2L(2mE)^{1/2}/\hbar} ,$$

or

$$ET > 2L(2mE)^{1/2} . \tag{15.7}$$

To satisfy these inequalities, we need only choose E large enough, for given T, L and m. A paradox, indeed!

We know that the Schrödinger equation allows arbitrary speeds; we cannot impose relativistic causality on nonrelativistic quantum mechanics. But this paradox does not involve superluminal speeds, or any speeds; like Bell's inequality, it involves only measurements at spacelike separations. Why isn't it consistent with relativistic causality?

Let us consider how large E has to be for the paradox to arise. Squaring both sides of Eq. (15.7), we have $ET^2 > 8L^2m > 8c^2T^2m$ or $E > 8mc^2$. We conclude that Bob can send Alice a superluminal signal if the binding energy of Alice's electron is several times mc^2.

The paradox, then, has a remarkable resolution. The quantum mechanics of a *single* particle in an arbitrary potential is inconsistent with the constraint of relativistic causality. Bob *can* send Alice a superluminal signal, unless we impose a (quite artificial) limit on the depth of Alice's potential well. But in relativistic quantum theory, particles multiply; a potential step greater than $2mc^2$ creates electron-positron pairs (as Klein's paradox [9] demonstrates). A potential step of $E > 2mc^2$ in Alice's well would create electron-positron pairs, and these would influence Alice's measurement much more than Bob's Geiger counter. Since, in relativistic quantum theory, Alice *cannot* bind a lone electron with a binding energy greater than $2mc^2$, Bob cannot send Alice a superluminal signal [10]. Indeed, this paradox belongs to the class of contradictions (or would if it had preceded the development of quantum field theory): it shows

that quantum mechanics with arbitrary potentials *and* fixed particle number is incompatible with relativistic causality. Only a new theory can resolve the paradox.

15.6 Towards Quantum Field Theory

For Schrödinger, $\Psi(\mathbf{x}, t)$ was a classical wave, analogous to a classical light wave. But for us, $\Psi(\mathbf{x}, t)$ is a quantum wave, and the analogy with classical light is not so relevant. More relevant is the analogy between $\Psi(\mathbf{x}, t)$ and quantum light. Like $\Psi(\mathbf{x}, t)$, quantum light waves are measurable. The quantum wave $\Psi(\mathbf{x}, t)$ describes a particle, and a quantum light wave describes particles – photons.

A flaw in the analogy is that $\Psi(\mathbf{x}, t)$ corresponds to only one particle, while a quantum light wave corresponds, in general, to many. But as the previous section indicates, this flaw disappears when $\Psi(\mathbf{x}, t)$ goes relativistic. Relativistic quantum theory is a many-particle theory. A nonrelativistic quantum wave can describe one particle; a relativistic quantum wave *must* describe many particles. Nonrelativistic quantum mechanics is self-consistent, but it is inconsistent with relativistic causality, because protective measurements – unlike quantum correlations, and unlike modular variables – allow Bob to send a superluminal signal to Alice. (See Chaps. 5–6.) But relativistic quantum theory must be consistent with relativistic causality; it cannot allow superluminal signalling. As the last section shows, there are two ways to prevent superluminal signalling: we can forbid any potential step that is several times mc^2, or we can allow the creation of particle-antiparticle pairs from the vacuum. There is no natural way to forbid deep potential wells; it would be unrealistic to forbid them. Thus, we must allow pair creation – which takes us to relativistic quantum field theory.

Problems

15.1 Do quantum fluctuations of the electromagnetic field resolve the paradoxes of Sect. 15.1? "No!" answers Simplicio. "Prof. Nu can indeed determine the position of a particle of charge Q by measuring its electric field. Although quantum fluctuations of the electric field reduce the precision of her field measurement, they are independent of Q. The larger Q, the larger the electric field that Prof. Nu measures, and the smaller the uncertainty in her determination of the position of the particle." What is wrong with Simplicio's answer?

*15.2 Apply Prob. 8.14 to resolve the paradox of Sect. 15.1. Assume that Prof. Nu measures the position of an electron via the impulse that its Coulomb field imparts to a test charge (either of the two charges in Nu's dipole) when the dipole is open, and calculate the influence of vacuum fluctuations of the electric field on the momentum of the electron and on its position as measured via the test charge.

15.3 Equations (15.1–2) define $\mathbf{J}_D(\mathbf{x}_0)$ and $\mathbf{J}_i(\mathbf{x})$. Compute the expectation value of $\mathbf{J}_D(\mathbf{x}_0)$ in the state $\Psi(\mathbf{x})$ and show that it equals $\int \mathbf{J}(\mathbf{x})\rho_D(\mathbf{x} - \mathbf{x}_0)d^3x$.

15.4 Consider a Hamiltonian $H_{int} = g(t)A_sP_d$ as a perturbation on a system prepared in its ground state.

(a) Assume $g(t) = 2T/(T^2+4\pi^2t^2)$ and show that, to first order in H_{int}, the probability of a transition to an excited state with energy E above the ground state is proportional to $e^{-2ET/h}$.
(b) Assume $g(t) = 1/T$ for $0 < t < T$ and $g(t) = 0$ at other times; show that the transition probability is proportional to $\sin^2(ET/2\hbar)/E^2T^2$.

*15.5 Let $g(t)$ be zero except for $0 \leq t \leq T$, with $\int_0^T |g(t)|dt$ finite and $\int_0^T g(t)dt = 1$. Define $G(E)$ to be the Fourier transform of $g(t)$:

$$G(E) = (2\pi\hbar)^{-1/2} \int_0^T g(t)e^{-iEt/\hbar}dt \ ,$$

where E is real. Assume that there is an $\epsilon > 0$ such that

$$|G(E)| \leq e^{-\epsilon|E|} \tag{15.8}$$

for all $|E|$ sufficiently large. Prove that $g(t)$ extends to an analytic function of complex t in the upper half-plane $\Im t \geq 0$. But $g(t)$, as defined, cannot be analytic! Hence the assumption is false; there is no $\epsilon > 0$ such that Eq. (15.8) holds for all $|E|$ sufficiently large.

*15.6 Section 15.5 presents Eq. (15.7) as a necessary and sufficient condition for a violation of relativistic causality. Sagredo now argues that Eq. (15.7) is not necessary. "Let Alice prepare particles in the ground state $|\Psi_0\rangle$ and measure $|\Psi_0(0)|$ on each particle over a time T. Alice's measurement yields a null result whenever it excites the particle and, additionally, whenever Bob looks for and finds the particle a distance L away. Whether or not Eq. (15.7) is satisfied, Bob can increase the probability of a null result – and thus send Alice a superluminal message – by looking for particles a distance L away!" However, there is a flaw in Sagredo's argument. Sagredo states that if Bob finds the particle, Alice's measurement yields $|\Psi_0(0)| = 0$. This statement is consistent with relativistic causality – the particle cannot reach Alice during her measurement if $L > cT$. Show, however, that the statement implies that Alice's measurement yields $|\Psi_0(0)| = 0$ with the same probability whether or not Bob looks for the particle a distance L away.

15.7 Let A, with eigenstates $|a_i\rangle$, be an observable of one system, and let $|\psi_i\rangle$ be states of another system, for $i = 1, \ldots, N$. Consider a protective measurement of A on the state

$$|\Psi\rangle = \sum_{i=1}^{N} c_i|a_i\rangle \otimes |\psi_i\rangle$$

of the two systems. Show [11] that the measurement yields tr (ρA), where ρ is the reduced density matrix

$$\rho = \sum_{i=1}^{N} |c_i|^2|a_i\rangle\langle a_i| \ .$$

Thus a protective measurement on a single system can yield tr (ρA). (See also Prob. 9.6.)

*15.8 Consider a Hamiltonian $H = \cos\phi\sigma_z + \sin\phi\sigma_x$ that changes adiabatically over a long time T, i.e. $d\phi/dt$ is at most of order $1/T$. The instantaneous eigenstates of H,

$$\psi_+ = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \end{pmatrix} , \quad \psi_- = \begin{pmatrix} -\sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{pmatrix} ,$$

have eigenvalues E_+ and E_-, respectively. Let $\psi(t)$ be the exact solution of the Schrödinger equation $i\hbar(d/dt)\psi(t) = H\psi(t)$; we can write $\psi(t)$ as

$$\psi(t) = \alpha_+(t)e^{-iE_+t/\hbar}\psi_+(t) + \alpha_-(t)e^{-iE_-t/\hbar}\psi_-(t) .$$

At $t = 0$, $\alpha_-(t)$ vanishes; at $t = T$, it can be chosen exponentially small in T (i.e. suppressed by $e^{-\epsilon T}$ where ϵ is a constant). Show, however, that $\alpha_-(t)$ cannot be exponentially small at all intermediate times.

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16 Weak Values

What is the problem with quantum mechanics? The problem is not just that quantum mechanics contradicts our everyday intuitions. So does the special theory of relativity! But special relativity offers us new and powerful intuitions about space and time, mass and energy, and causality. By contrast, quantum mechanics seems to offer only negatives: no determinism, no complete description, no objective reality. What about the positive in quantum mechanics? In these last three chapters, we emphasize the positive; and we emphasize that the positive in quantum mechanics often hides behind the negative.

For example, the word "quantum" entered physics in 1900 as a constraint on classically allowed values of observables. Planck's quantum postulate constrains the energies of a resonator to discrete values. And in quantum mechanics, it is axiomatic that the only allowed values of an observable are its eigenvalues. Yet in many ways, quantum mechanics is *freer* from constraints than classical mechanics. Take quantum tunnelling as an example. In classical mechanics, a particle cannot cross a potential barrier greater than its total energy; it cannot have negative kinetic energy. In quantum mechanics, it *can* cross such a barrier. Can it have negative kinetic energy? The axiomatic answer is "No! The eigenvalues of kinetic energy are all positive!" (See Prob. 16.1.) But Sect. 16.4 goes beyond this answer to define the *weak* value of an observable. The weak value of an observable need not be an eigenvalue; indeed, it need not be any classically allowed value. Weak values offer intuition about a quantum world that is freer than we imagined – a world in which particles travel faster than light, carry unbounded spin, and have negative kinetic energy.

16.1 A Weak Measurement

Here is a thought experiment that seems to disobey the uncertainty relations. It disobeys very, very infrequently. But even if it disobeys only *once*, quantum mechanics is inconsistent. So this thought experiment, like the thought experiments of Sects. 2.4, 4.1 and 15.1, challenges the consistency of quantum mechanics.

Consider a large number N of spin-1/2 particles in the initial spin state $|\Psi_{in}\rangle$:

$$|\Psi_{in}\rangle = 2^{-N/2} \bigotimes_{i=1}^{N} (|\uparrow\rangle_i + |\downarrow\rangle_i) \ .$$

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Here $S_z^{(i)}|\uparrow\rangle_i = \hbar/2$, $S_z^{(i)}|\downarrow\rangle_i = -\hbar/2$. Let S_z denote the z-component of total spin:

$$S_z = \sum_{i=1}^{N} S_z^{(i)} \ .$$

For a measurement of S_z, we take the interaction Hamiltonian

$$H_{int} = g(t) P_d S_z/\sqrt{N} \ ;$$

as usual, P_d is conjugate to the position Q_d of a pointer, and $\int dt g(t) = g_0$. The factor $1/\sqrt{N}$ weakens the effective coupling to each spin. We assume that the measurement is impulsive; then the change in Q_d during the measurement is $g_0 S_z/\sqrt{N}$. The uncertainty in this measurement of S_z is $\Delta S_z = (\sqrt{N}/g_0)\Delta Q_d$, where ΔQ_d is the uncertainty in Q_d.

The expectation value of S_z in the state $|\Psi_{in}\rangle$ vanishes, while the expectation value of S_z^2 is $N\hbar^2/4$. Hence most measured values of S_z are of order $\sqrt{N}\hbar/2$. Such a value may well be an error, since ΔS_z is itself proportional to $\sqrt{N}$. But suppose a measured value of S_z is $N\hbar/2$. Is it an error?

We expect a measurement of S_z to change $|\Psi_{in}\rangle$, which is not an eigenstate of S_z. How does $|\Psi_{in}\rangle$ change? An individual spin state $(|\uparrow\rangle_i + |\downarrow\rangle_i)/\sqrt{2}$ evolves to

$$[e^{-ig_0 P_d/2\sqrt{N}}|\uparrow\rangle_i + e^{ig_0 P_d/2\sqrt{N}}|\downarrow\rangle_i]/\sqrt{2} \ .$$

Hence the probability that the measurement leaves the i-th spin in the initial state $(|\uparrow\rangle_i + |\downarrow\rangle_i)/\sqrt{2}$ is

$$\langle\Phi_d|\cos^2(g_0 P_d/2\sqrt{N})|\Phi_d\rangle \ ,$$

where $\Phi_d(Q_d)$ represents the initial wave function of the pointer. The probability that the measurement leaves *all* the spins in the initial state $|\Psi_{in}\rangle$ is $\langle\Phi_d|\cos^{2N}(g_0 P_d/2\sqrt{N})|\Phi_d\rangle$, which approaches $\langle\Phi_d|e^{-g_0^2 P_d^2/4}|\Phi_d\rangle$ for large N. We can estimate the expectation value of P_d^2 from $(\Delta P_d)^2 \approx \hbar^2/4(\Delta Q_d)^2$. (The expectation value of P_d vanishes for the stationary pointer.) Thus for large N, the measurement leaves the initial state $|\Psi_{in}\rangle$ unchanged with probability

$$e^{-g_0^2\hbar^2/16(\Delta Q_d)^2}$$

which can be arbitrarily close to 1.

So it seems that a measurement could yield $S_z = N\hbar/2$ without changing the initial state $|\Psi_{in}\rangle$ at all. But $|\Psi_{in}\rangle$ is an eigenstate of S_x, not of S_z. Can we know both S_z and S_x at the same time? How can we explain this measurement?

These formulas apply also to a two-slit interference experiment with N particles. Let the states $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ represent the i-th particle passing through the left and right slits, respectively. (See Fig. 16.1.) Then measuring S_z shows the number of particles passing through the left slit minus the number passing through the right slit. For large enough N, a measurement of S_z on the state $|\Psi_{in}\rangle$ could show all the particles passing through the left slit, with no change in $|\Psi_{in}\rangle$. But with no change in $|\Psi_{in}\rangle$, the measurement does not destroy the interference!

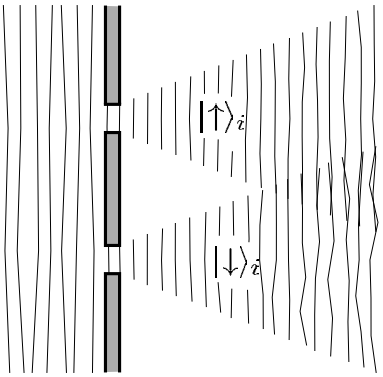

Figure 16.1: The paradox of Sect. 16.1 formulated as a two-slit interference experiment. The states $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ represent the i-th particle passing through the upper and lower slit, respectively.

16.2 A Paradox of Errors

In this book, measurements are quantum measurements, i.e. we treat measuring devices as quantum systems. Chapter 7 defines an interaction Hamiltonian for measuring an observable A_s:

$$H_{int} = g(t)A_sP_d \ . \tag{16.1}$$

It is convenient to take $g(t)$ impulsive and $g_0 = \int g(t)dt = 1$. The measuring device is a quantum system, hence it has a quantum state. But Chaps. 7 and 8, which discuss quantum measurements in the Heisenberg formalism, do not refer to the state of the measuring device; and even Chap. 9, which discusses quantum measurements in the Schrödinger formalism, refers to the state of the measuring device only formally. For example, Eq. (9.3) refers to the initial state $|0\rangle$ of a pointer without further defining $|0\rangle$. In this respect, our treatment of quantum measurements is not yet complete. When we discuss the state of the measuring device, we see how the model of von Neumann accounts for errors in quantum measurements.

Let us define an initial state $\Phi_{in}(Q_d)$ of the measuring device:

$$\Phi_{in}(Q_d) = (\epsilon^2\pi)^{-1/4}e^{-Q_d^2/2\epsilon^2} \ . \tag{16.2}$$

The width of the peak in $\Phi_{in}(Q_d)$ depends on ϵ according to $\Delta Q_d = \epsilon/\sqrt{2}$. Small ϵ corresponds to small uncertainty in Q_d and in A_s. But it also corresponds to a strong measurement interaction: since ΔP_d is large, we cannot take the range of P_d in Eq. (16.1) to be small. Conversely, large ϵ corresponds to large uncertainty in Q_d and A_s, and also to a weak measurement interaction. The limit $\epsilon \to 0$ defines an *ideal* measurement; in this limit ΔQ_d vanishes, and the change in the pointer position equals an eigenvalue of A_s with no error. Real measurements, however, can yield errors; and $\Phi_{in}(Q_d)$ is a source of errors. Note that a quantum measurement of A_s obeying Eqs. (16.1–2) could yield *any* value, although large errors are exponentially suppressed. There is no mystery in these errors; since ΔQ_d does not vanish, we expect them. Measurements of a nonnegative observable such as kinetic energy could even yield unphysical negative values. Of course, the Q_d dial could have a pin to prevent the pointer from turning to negative values, but let us assume it does not. It should not. For even if a measured value of

A_s is unphysical, it belongs to a physical distribution of measured values. It helps us estimate the width and peak value of the distribution, and we should not throw it out.

So just by defining the initial state of the measuring device, we obtain a model for measurement errors. In this model, errors have a natural interpretation: they represent scatter around the true value of an observable, which can only be an eigenvalue of that observable. They are due not to the measured system, but to uncertainty in the measuring device. If a measured system is in an eigenstate of A_s, then ΔA_s depends *only* on ΔQ_d, and the model implies (as it should) that more accurate measurements (smaller ΔQ_d) yield fewer errors (smaller ΔA_s). Indeed, since ΔA_s depends only on ΔQ_d, errors tell us nothing about the measured system. For the example of kinetic energy measurements, the model implies that measured values may be negative, but also that negative values are unphysical – errors due to the uncertainty in Q_d.

At the risk of belaboring the obvious, we stress that this interpretation of measurement errors is natural and consistent. Yet we now present a thought experiment [1] with an incompatible interpretation. Here is the experiment in brief: We prepare a large number of particles bound in a potential well, in an eigenstate of energy. Next, we measure the kinetic energy of each particle, with an accuracy that depends on the measuring device. Finally, we look for particles far from the potential well. That is, after the kinetic energy measurement is complete, we measure the positions of all the particles, and select particles found "far enough" from the potential well – with "far enough" depending on the accuracy of the kinetic energy measurement. We find very few particles "far enough" from the potential well, but for these particles, we find that the kinetic energy measurements gave *negative* values. Even more remarkable is the distribution of the values: the peak value equals the negative kinetic energy (total energy minus potential energy) of a bound particle outside its potential well; also, the width of the peak is the expected width for measurements with this measuring device. So are these values errors?

Here is the thought experiment in detail: The Hamiltonian for a bound particle in one dimension is

$$H_s = \frac{p^2}{2m} + V(x) \; . \tag{16.3}$$

For simplicity, we choose $V(x)$ to be a negative δ-function potential, vanishing for $x \neq 0$. Then H_s has a single bound state,

$$\Psi_{in}(x) = \sqrt{\alpha} e^{-\alpha|x|} \; ,$$

with eigenvalue $-\hbar^2\alpha^2/2m$. Let $\Psi_{in}(x)$ be the initial state of the particle. The initial state of the measuring device is the gaussian $\Phi_{in}(Q_d)$ of Eq. (16.2), and the interaction Hamiltonian for the measurement is H_{int} of Eq. (16.1) with $p^2/2m$ for A_s. (Nonrelativistic quantum mechanics is consistent with such a nonlocal interaction.) For simplicity, again, we assume that the measurement is impulsive, so that we can neglect H_s for the duration of the measurement. Then the combined state of the particle and measuring device evolves from $\Psi_{in}(x)\Phi_{in}(Q_d)$ at the beginning of the measurement to

$$e^{-(i/\hbar)P_d p^2/2m}\Psi_{in}(x)\Phi_{in}(Q_d) \tag{16.4}$$

at the end. At the end of the measurement, we look for the particle far from the origin. Let the final state of the particle be

$$\Psi_{fin}(x) = (\delta^2\pi)^{-1/4} e^{-(x-x_0)^2/2\delta^2} \ ,$$

where δ and x_0 satisfy two conditions:

$$\delta > \alpha\hbar^2/m\epsilon \tag{16.5}$$

and

$$x_0 \gg \delta^3 m\epsilon/\hbar^2 \ . \tag{16.6}$$

In words, for a more accurate (smaller ϵ) measurement of kinetic energy, the final state of the particle must be broader (larger δ) and farther from the well (larger x_0). (Together, Eqs. (16.5–6) imply $x_0 \gg \alpha^3\hbar^4/m^2\epsilon^2$.) Of course, there is no assurance that we will find the particle in this final state; on the contrary, the probability is small. But we assume that we can repeat the experiment as many times as we like, until a sufficient number of particles satisfy these conditions.

Now for these particles only, we look at the values of kinetic energy that the measuring device recorded before we checked the final position of each particle. What do the measurements show? We obtain the final state $\Phi_{fin}(Q_d)$ of the measuring device by projecting Eq. (16.4) onto $\Psi_{fin}(x)$:

$$\Phi_{fin}(Q_d) = N\langle\Psi_{fin}|e^{-(i/\hbar)P_d p^2/2m}|\Psi_{in}\rangle\Phi_{in}(Q_d) \ . \tag{16.7}$$

Here N is a normalization factor. Equation (16.7) entails an integration over x. Since $\Psi_{fin}(x)$ is localized far from the origin, it is plausible that the operator p in Eq. (16.7) takes the value $-i\hbar\alpha$ when applied to $\Psi_{in}(x)$. If so, the final, normalized state of the measuring device is

$$\Phi_{fin}(Q_d) = \Phi_{in}(Q_d + \alpha^2\hbar^2/2m) \ , \tag{16.8}$$

and represents the measuring device with its pointer shifted to the "unphysical" negative value $-\alpha^2\hbar^2/2m$.

To derive Eq. (16.8) rigorously, we express $\Psi_{in}(x)$ as a Fourier transform,

$$\Psi_{in}(x) = \int_{-\infty}^{\infty} dp \frac{e^{-ipx/\hbar}}{\alpha^2\hbar^2 + p^2}$$

up to a normalization factor, and replace the operator p with its eigenvalues. The exponential of $-(i/\hbar)P_d p^2/2m$ translates Q_d to $Q_d - p^2/2m$, and after the x-integration Eq. (16.7) becomes

$$\Phi_{fin}(Q_d) = N' \int_{-\infty}^{\infty} dp \frac{e^{-ipx_0/\hbar - \delta^2 p^2/2\hbar^2}}{\alpha^2\hbar^2 + p^2} \Phi_{in}(Q_d - p^2/2m) \ , \tag{16.9}$$

where N' is for normalization. The integrand in Eq. (16.9) has poles at $p = \pm i\alpha\hbar$, and we can regard the integral as part of a contour integral. (See Fig. 16.2.) The contour includes the line $\Im p = -p_0$ with $p_0 > \alpha\hbar$. Then the integral reduces to two terms: a pole term proportional to

$$\Phi_{in}(Q + \alpha^2\hbar^2/2m) \ ,$$

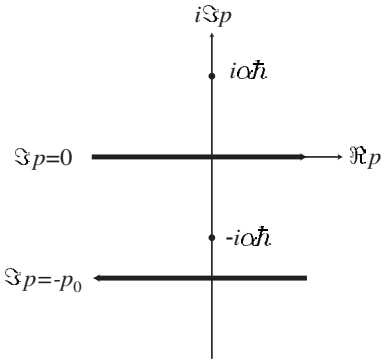

Figure 16.2: The contour for the integrations in Eqs. (16.9–10).

and a correction, the integral in Eq. (16.9) with $p - ip_0$ replacing p:

$$\frac{\hbar\alpha}{\pi} e^{\alpha x_0 - \alpha^2\delta^2/2} \int_{-\infty}^{\infty} dp \frac{e^{-(p-ip_0)^2\delta^2/2\hbar^2 - i(p-ip_0)x_0/\hbar}}{\alpha^2\hbar^2 + (p-ip_0)^2} \Phi_{in}\left[Q_d - (p-ip_0)^2/2m\right] . \tag{16.10}$$

The absolute value of Eq. (16.10) depends on x_0 only through the exponential of $(\alpha - p_0/\hbar)x_0$. The rest of Eq. (16.10) is finite. Then, since $\alpha - p_0/\hbar$ is negative, the correction vanishes in the limit $x_0 \to \infty$. The correction is small if x_0 and δ satisfy Eqs. (16.5–6). (See Prob. 16.4.)

So we arrive at a paradox. On the one hand, negative measured values of kinetic energy arise from uncertainty in the initial position of the pointer, and not from the measured system. Therefore they are unphysical. On the other hand, the negative measured values of kinetic energy in our thought experiment are uncannily physical. Does a particle with total energy $-\alpha^2\hbar^2/2m$, in a region of vanishing potential energy, have kinetic energy equal to $-\alpha^2\hbar^2/2m$? Indeed, kinetic energy measurements of such particles, found far from the potential well that binds them, yield a distribution peaked at $-\alpha^2\hbar^2/2m$; and the width of the peak corresponds to the accuracy of the measuring device. We find this peak for any realistic measurements, i.e. for any ϵ greater than zero.

Freud found hidden meaning in slips of the tongue; here, we find hidden meaning in measurement errors. But now measurement errors have two incompatible interpretations!

16.3 Pre- and Postselected Ensembles

In classical mechanics, the initial state of a closed system determines its final state, and vice versa. For a complete description of a classical system, we can impose a complete set of initial *or* final boundary conditions; complete sets of initial *and* final conditions would be either redundant or inconsistent. Not so in quantum mechanics; the initial state of a closed system does not determine its final state, and we can impose complete initial *and* final boundary conditions. The contrast between classical and quantum boundary conditions suggests that, for the most complete description of a *quantum* system, we not only *can*, we *must* impose complete sets of initial and final boundary conditions.

How do we impose final boundary conditions? The same way we impose initial boundary conditions: we select for them. (See Sect. 10.4.) For example, we impose the initial boundary condition $S_x = \hbar/2$ by measuring S_x on spin-1/2 particles and selecting those for which $S_x = \hbar/2$. Similarly, we can impose the final boundary condition $S_z = \hbar/2$ by measuring S_z on spin-1/2 particles and selecting those for which $S_z = \hbar/2$. Either way we select, but for an initial boundary condition we *pre*select (before other measurements), while for a final boundary condition we *post*select (after other measurements). Together, initial and final boundary conditions define a pre- and postselected (PPS) ensemble. PPS ensembles arise naturally in sequences of measurements. Given a sequence of measurements, we define a PPS ensemble by preselecting a result of the initial measurement and postselecting a result of the final measurement. Intermediate measurements are measurements *on* the PPS ensemble.

If these intermediate measurements are ideal – if the measuring devices do not introduce errors – then we can apply the ABL formula to them. (See Sect. 10.3.) But in any realistic measurement there is a chance of errors arising from the measuring device. Can we neglect the chance of errors? If not, we cannot apply the ABL formula. The thought experiment of the last section is an example of a realistic measurement on a PPS ensemble. Here we encounter another example, and the next section treats such measurements in general.

The example [2] concerns a PPS ensemble of spin-1/2 particles. The initial boundary condition $S_x = \hbar/2$ and final boundary condition $(S_z - S_x)/\sqrt{2} = \hbar/2$ together define the ensemble. Suppose we measure S_z on this ensemble. The interaction Hamiltonian for each particle is Eq. (16.1) with S_z in place of A_s:

$$H_{int}(t) = g(t) S_z P_d \ .$$

We assume, for simplicity, that there are no other terms in the Hamiltonian; then S_z does not change during the measurement. We define the initial state $\Phi_{in}(Q_d)$ of the measuring device as in Eq. (16.2). Thus the initial state of one spin-1/2 particle and a measuring device is

$$\frac{1}{\sqrt{2}} \left(|\uparrow\rangle + |\downarrow\rangle \right) \Phi_{in}(Q_d)$$

and it evolves during the S_z-measurement to

$$\begin{aligned} &\frac{1}{\sqrt{2}} e^{-(i/\hbar) S_z P_d} \left(|\uparrow\rangle + |\downarrow\rangle \right) \Phi_{in}(Q_d) \\ &\quad = (4\epsilon^2 \pi)^{-1/4} \left(|\uparrow\rangle e^{-(Q_d - \hbar/2)^2/2\epsilon^2} + |\downarrow\rangle e^{-(Q_d + \hbar/2)^2/2\epsilon^2} \right) \ . \end{aligned} \tag{16.11}$$

Finally, we postselect the state

$$\frac{1}{2}(2+\sqrt{2})^{1/2} |\uparrow\rangle - \frac{1}{2}(2-\sqrt{2})^{1/2} |\downarrow\rangle \ ,$$

which is the eigenstate of $(S_z - S_x)/\sqrt{2}$ with eigenvalue $\hbar/2$. Projecting Eq. (16.11) onto this eigenstate, we obtain the (unnormalized) state of the measuring device at the end of the experiment:

$$\frac{1}{2}(2+\sqrt{2})^{1/2} e^{-(Q_d - \hbar/2)^2/2\epsilon^2} - \frac{1}{2}(2-\sqrt{2})^{1/2} e^{-(Q_d + \hbar/2)^2/2\epsilon^2} \ . \tag{16.12}$$

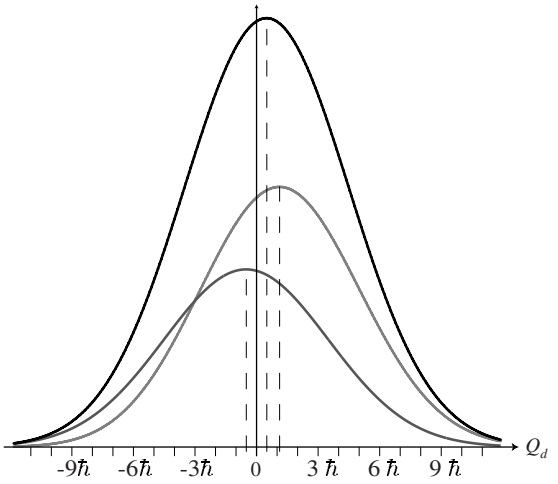

Figure 16.3: Graph of the two gaussian functions in Eq. (16.12), and their difference (peaked at $Q_d \approx 1.1\hbar$), for $\epsilon = 4\hbar$.

In the limit $\epsilon \to 0$, we find the pointer in a superposition of two readings; with probability $(2+\sqrt{2})/4$, the pointer reads $\hbar/2$, and with probability $(2-\sqrt{2})/4$, it reads $-\hbar/2$. These probabilities are in accord with the ABL formula. But let us consider *large* ϵ. From Eq. (16.12) we obtain

$$e^{-Q_d^2/2\epsilon^2}\left[\frac{1}{2}(2+\sqrt{2})^{1/2}e^{Q_d\hbar/2\epsilon^2} - \frac{1}{2}(2-\sqrt{2})^{1/2}e^{-Q_d\hbar/2\epsilon^2}\right]$$

as the state of the measuring device (up to normalization). We expand the exponential terms in the brackets:

$$\begin{aligned}
&\frac{1}{2}(2+\sqrt{2})^{1/2}e^{Q_d\hbar/2\epsilon^2} - \frac{1}{2}(2-\sqrt{2})^{1/2}e^{-Q_d\hbar/2\epsilon^2} \\
&\quad\approx \frac{1}{2}(2+\sqrt{2})^{1/2}\left(1+\frac{Q_d\hbar}{2\epsilon^2}\right) - \frac{1}{2}(2-\sqrt{2})^{1/2}\left(1-\frac{Q_d\hbar}{2\epsilon^2}\right) \\
&\quad= \frac{\sqrt{2}}{2}(2-\sqrt{2})^{1/2}\left(1+[1+\sqrt{2}]\frac{Q_d\hbar}{2\epsilon^2}\right) \\
&\quad\approx \frac{\sqrt{2}}{2}(2-\sqrt{2})^{1/2}e^{(1+\sqrt{2})Q_d\hbar/2\epsilon^2}\ .
\end{aligned}$$

Hence the final normalized state of the measuring device, for large ϵ, is

$$\begin{aligned}
\Phi_{fin}(Q_d) &\approx (\epsilon^2\pi)^{-1/4}e^{-(Q_d-[1+\sqrt{2}]\hbar/2)^2/2\epsilon^2} \\
&= \Phi_{in}(Q_d - [1+\sqrt{2}]\hbar/2)\ .
\end{aligned}$$

If all the particles in the PPS ensemble couple to pointers, the distribution of final pointer positions approaches $|\Phi_{fin}(Q_d)|^2$, shown in Fig. 16.3.

Wait! We made an error!

But we made no error. The distribution really peaks at the value $(1+\sqrt{2})\hbar/2$, more than twice the largest eigenvalue of S_z. Moreover, the width of the peak corresponds to the accuracy of the measuring devices. The measuring devices "err" so consistently, they seem to be *trying* to err.

In Hilbert space no one can hear you scream; but these mute measuring devices are trying to tell a story. What is their story?

16.4 Weak Measurements and Weak Values

We can generalize the example of the last section. For a pre- and postselected ensemble, we preselect an initial state $|\Psi_{in}\rangle$ and postselect a final state $|\Psi_{fin}\rangle$, with $\langle\Psi_{fin}|\Psi_{in}\rangle \neq 0$. In between the pre- and postselection, we measure a nondegenerate Hermitian operator. The initial state of the measuring device is $\Phi_{in}(Q_d)$ in Eq. (16.2), and the interaction Hamiltonian for the measurement is H_{int} in Eq. (16.1). (We now drop the subscript on A_s.) Again, we choose $g(t)$ impulsive and assume that, during the measurement, the time evolution operator for the measured system and measuring device is

$$e^{-(i/\hbar)AP_d} .$$

Then after the postselection, the (unnormalized) state of the measuring device is

$$\langle\Psi_{fin}|e^{-(i/\hbar)AP_d}|\Psi_{in}\rangle\Phi_{in}(Q_d) = \sum_i \langle\Psi_{fin}|a_i\rangle\langle a_i|\Psi_{in}\rangle\Phi_{in}(Q_d - a_i) .$$

Here the a_i are eigenvalues of A and the $|a_i\rangle$ are corresponding eigenstates.

In the limit $\epsilon \to 0$, $\Phi_{in}(Q_d - a_i)$ and $\Phi_{in}(Q_d - a_j)$ are orthogonal for $i \neq j$. So for ideal measurements, the probability of an outcome a_i is proportional to the norm of $\langle\Psi_{fin}|a_i\rangle\langle a_i|\Psi_{in}\rangle$ squared, as the ABL formula states. (See Eq. (10.8).) In the opposite limit, ϵ is large and so is the uncertainty in Q_d. On the other hand, ΔP_d is small,

$$\Delta P_d = \hbar/\sqrt{2}\epsilon ,$$

and the expectation value of P_d vanishes for a stationary pointer. Thus we take P_d to be small and expand the time evolution operator:

$$\begin{aligned}\langle\Psi_{fin}|e^{-(i/\hbar)AP_d}|\Psi_{in}\rangle\Phi_{in}(Q_d) &\approx \langle\Psi_{fin}|1 - (i/\hbar)AP_d|\Psi_{in}\rangle\Phi_{in}(Q_d) \\ &= \langle\Psi_{fin}|\Psi_{in}\rangle\left[1 - (i/\hbar)\langle A\rangle_w P_d\right]\Phi_{in}(Q_d) \\ &\approx \langle\Psi_{fin}|\Psi_{in}\rangle e^{-(i/\hbar)\langle A\rangle_w P_d}\Phi_{in}(Q_d) . \qquad (16.13)\end{aligned}$$

Here $\langle A\rangle_w$ (also denoted A_w) is the *weak* value of A on the PPS ensemble [3]:

$$\langle A\rangle_w \equiv \frac{\langle\Psi_{fin}|A|\Psi_{in}\rangle}{\langle\Psi_{fin}|\Psi_{in}\rangle} . \qquad (16.14)$$

The notation $\langle A\rangle_w$ recalls $\langle A\rangle$; weak values indeed reduce to expectation values when $|\Psi_{fin}\rangle = |\Psi_{in}\rangle$. Eq. (16.13) shows that the displacement of the pointer is $\langle A\rangle_w$:

$$\Phi_{fin}(Q_d) \approx e^{-(i/\hbar)\langle A\rangle_w P_d}\Phi_{in}(Q_d) = \Phi_{in}(Q_d - \langle A\rangle_w) \ .$$

Thus for large ϵ, the measured values of A cluster around $\langle A\rangle_w$.

The surprising value $S_z = (1+\sqrt{2})\hbar/2$ of the last section is a weak value, as we check by substituting $|\Psi_{in}\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and $|\Psi_{fin}\rangle = \frac{1}{2}(2+\sqrt{2})^{1/2}|\uparrow\rangle - \frac{1}{2}(2-\sqrt{2})^{1/2}|\downarrow\rangle$ into Eq. (16.14). So is the negative value of kinetic energy of Sect. 16.2; we have

$$\langle p^2/2m\rangle_w = \frac{\langle\Psi_{fin}|(p^2/2m)|\Psi_{in}\rangle}{\langle\Psi_{fin}|\Psi_{in}\rangle} \ ,$$

where $|\Psi_{in}\rangle$ represents a particle bound by the δ-function potential well, and $|\Psi_{fin}\rangle$ represents a particle far from the well. Equation (16.3) implies

$$p^2/2m = H_s - V(x)$$

as an operator equation, and Eq. (16.14) implies, in general,

$$\langle A+B\rangle_w = \langle A\rangle_w + \langle B\rangle_w \ ;$$

and in particular,

$$\langle p^2/2m\rangle_w = \langle H_s\rangle_w - \langle V(x)\rangle_w \ .$$

Now $\langle H_s\rangle_w = -\alpha^2\hbar^2/2m$ because $|\Psi_{in}\rangle$ is an eigenstate of H_s,

$$H_s|\Psi_{in}\rangle = -(\alpha^2\hbar^2/2m)|\Psi_{in}\rangle \ ,$$

and $\langle V(x)\rangle_w = 0$ because $V(x)$ vanishes in the final state, $V(x)|\Psi_{fin}\rangle \approx 0$. We have $\langle p^2/2m\rangle_w = -\alpha^2\hbar^2/2m$, the "unphysical" negative kinetic energy of Sect. 16.2.

A measurement of A that yields $\langle A\rangle_w$ consistently is a *weak* measurement. As Eq. (16.13) shows, a measurement of A on the PPS ensemble always yields $\langle A\rangle_w$ if ϵ is sufficiently large. (See also Prob. 16.6.) But from the example of negative kinetic energy we see that a measurement may be weak even if ϵ is not large. For any given ϵ, the kinetic energy measurement is weak if $|\Psi_{fin}\rangle$ satisfies Eqs. (16.5–6).

In the other example, $|\Psi_{fin}\rangle$ is given and the measurement yields $\langle S_z\rangle_w = (1+\sqrt{2})\hbar/2$ only for large ϵ. But suppose we replace the single spin-1/2 particle with N spin-1/2 particles. Let S_x, S_y and S_z represent components of total spin:

$$S_x = \sum_{i=1}^{N} S_x^{(i)} \ , \quad S_y = \sum_{i=1}^{N} S_y^{(i)} \ , \quad S_z = \sum_{i=1}^{N} S_z^{(i)} \ , \tag{16.15}$$

where i indexes the N particles. The boundary conditions

$$\frac{S_x}{N}|\Psi_{in}\rangle = \frac{\hbar}{2}|\Psi_{in}\rangle \ , \quad \frac{S_z - S_x}{\sqrt{2}N}|\Psi_{fin}\rangle = \frac{\hbar}{2}|\Psi_{fin}\rangle$$

define a PPS ensemble. On this PPS ensemble we measure the observable $A = S_z/N$, which has eigenvalues between $-\hbar/2$ and $\hbar/2$. Writing

$$A = \frac{S_x}{N} + \sqrt{2} \cdot \frac{S_z - S_x}{\sqrt{2}N} \ , \tag{16.16}$$

we immediately obtain $\langle A \rangle_w = (1 + \sqrt{2})\hbar/2$. Now for any given ϵ, a measurement of A is weak for large enough N. (See Sect. 16.6.)

Experiments on an optical version of this example demonstrated weak values [4]. Other applications of weak values include measurements of arrival times of particles tunnelling through potential barriers [5], loss of interference in two-slit interferometers [6], and measurements of superluminal and subluminal photon speeds [7].

16.5 A Quantum Shell Game

Place three walnut shells on a table, and let's play a "shell game".[1] One player – call him Shelly – shifts a small coin quickly among the three shells, while the rest of us guess (and bet) where it is. In a (ho hum) classical shell game, the coin is certainly under one of the shells, and Shelly, at least, knows which one. What about a quantum shell game?

A quantum coin can be in a superposition of three states $|A\rangle$, $|B\rangle$, $|C\rangle$ – one state for each shell. For example, Shelly could prepare the coin in the initial state

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle) \ . \tag{16.17}$$

By looking under one of the shells, we measure one of the projection operators $\mathbf{\Pi}_A = |A\rangle\langle A|$, $\mathbf{\Pi}_B = |B\rangle\langle B|$, $\mathbf{\Pi}_C = |C\rangle\langle C|$. The result of the measurement is either 1 or 0: the coin is either there or not there; it is never partly there. To model a quantum measurement of a projection operator, let's suppose that the coin carries a positive charge. We send an electron past a shell, and if the coin is in the shell, its charge deflects the electron. A model interaction Hamiltonian for a measurement of $\mathbf{\Pi}_C$ is

$$H_{int} = -g(t)\mathbf{\Pi}_C X \ , \tag{16.18}$$

where X is the transverse position of the electron. Again, it is convenient to assume $g_0 = \int g(t)dt = 1$. Let the initial state of the electron be

$$\Phi(P) = (\epsilon^2\hbar^2\pi)^{-1/4} e^{-P^2/2\epsilon^2\hbar^2} \ , \tag{16.19}$$

where P is the transverse momentum; for simplicity, we neglect the other components of the electron momentum. The transverse momentum of the electron changes according to

$$\frac{dP}{dt} = g(t)\mathbf{\Pi}_C \ ,$$

so it changes by $\mathbf{\Pi}_C$ during the measurement. If the coin is in shell C (in state $|C\rangle$) the electron deflects to the right; if it is not in shell C, the electron continues in a straight line. (See Fig. 16.4.)

[1] Also called "thimblerig".

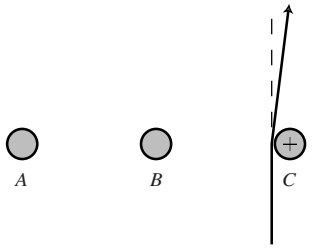

Figure 16.4: An electron deflected by positive charge in shell C.

So far, the quantum game resembles the classical game. But Shelly plays the game with pre- *and* postselection. So if our measurement is ideal, we can apply the ABL formula; if our measurement is weak, we can calculate weak values of $\mathbf{\Pi}_A$, $\mathbf{\Pi}_B$ and $\mathbf{\Pi}_C$. Suppose the final state is

$$|\Psi_{fin}\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle - |C\rangle) \ .$$

The ABL formula shows that the coin is certainly in shell A if we look there:

$$\frac{|\langle\Psi_{fin}|\mathbf{\Pi}_A|\Psi_{in}\rangle|^2}{|\langle\Psi_{fin}|\mathbf{\Pi}_A|\Psi_{in}\rangle|^2 + |\langle\Psi_{fin}|(\mathbf{\Pi}_B + \mathbf{\Pi}_C)|\Psi_{in}\rangle|^2} = 1 \ .$$

(See Prob. 10.6.) Similarly, the coin is certainly in shell B if we look *there*. Only if we look in shell C do we have a chance (equal to 4/5) of not finding the coin. So – if Shelly postselects this final state – the quantum shell game pays better than the classical game!

Similarly, the weak values of $\mathbf{\Pi}_A$ and $\mathbf{\Pi}_B$ are both 1. But the weak value of $\mathbf{\Pi}_C$ is -1. What is the meaning of $\langle\mathbf{\Pi}_C\rangle_w = -1$? Measurement supplies the meaning: Eqs. (16.13) and (16.18) imply that the electron we send past shell C deflects to the *left* instead of to the right, as if the coin carried a *negative* charge. (See Fig. 16.5.) But $\mathbf{\Pi}_C$ has no negative eigenvalue! An optical version of this experiment, in which the three "shells" were three interfering photon paths in a modified Mach-Zehnder interferometer, indeed yielded these weak values [8].

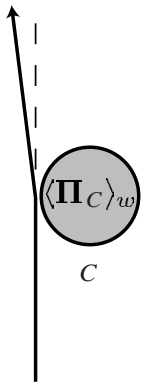

Figure 16.5: An electron deflected by weak negative charge in shell C.

The weak value of $\mathbf{\Pi}_C$ might not even be real. In general, Eq. (16.14) is complex. Suppose the initial state is still $|\Psi_{in}\rangle$ of Eq. (16.17) but Shelly postselects the final state

$$|\Psi_{fin}\rangle = \frac{1}{\sqrt{3}}(i|A\rangle - |B\rangle - i|C\rangle) \ .$$

For this PPS ensemble the weak value of $\mathbf{\Pi}_B$ is 1, while $\langle\mathbf{\Pi}_A\rangle_w = -i$ and $\langle\mathbf{\Pi}_C\rangle_w = i$. Negative charge is familiar, but what about imaginary charge? How do we measure – and interpret – an imaginary weak value?

Imaginary weak values have a general effect on the size of PPS ensembles [9]. Consider a weak measurement of an observable A on an ensemble of systems preselected in the state $|\Psi_{in}\rangle$ and postselected in the state $|\Psi_{fin}\rangle$. What fraction of the *pre*selected systems pass the *post*selection? Equation (16.13) shows that, as long as $\langle A\rangle_w$ is real, the fraction is $|\langle\Psi_{fin}|\Psi_{in}\rangle|^2$. That is, the size of the PPS ensemble depends on $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$ – but not on A. But if $\langle A\rangle_w$ is not real, then $e^{-(i/\hbar)\langle A\rangle_w P_d}$ is not unitary, and the size of the PPS ensemble depends on the imaginary part of $\langle A\rangle_w$. That is, the measurement creates or destroys systems in the pre- and postselected states. So $\langle A\rangle_w P_d$ imitates a nonunitary Hamiltonian, i.e. an effective Hamiltonian for a process that creates or destroys systems. (See Prob. 16.9.) In our example, when $\langle\mathbf{\Pi}_C\rangle_w$ is imaginary $\langle\mathbf{\Pi}_C\rangle_w X$ imitates an effective Hamiltonian that couples an imaginary charge to the passing electron.[2]

16.6 The Quantum Walk

A weak value depends on pre- and postselected states of a system. Hence a measured weak value depends on the pre- and postselected states of a system. A measured value depends on the state of a system *after* the measurement? Can we say that? The most complete description of a quantum system, we claim, involves *two* state vectors, past and future boundary conditions on the system. The contrary claim is that the most complete description of a quantum system involves only one state, and we are just playing a game of errors: Weak measurements yield errors, from which we postselect weak values.

A game – even a game of errors – has rules. Consider a system with a Hamiltonian H and two observables, A and B, that commute with H but not with each other. The initial state of the system is $|\Psi_{in}\rangle$, with $A|\Psi_{in}\rangle = a|\Psi_{in}\rangle$; the final state of the system is $|\Psi_{fin}\rangle$, with $B|\Psi_{fin}\rangle = b|\Psi_{fin}\rangle$. What can we say about possible intermediate measurements? We can say that a measurement of A at an intermediate time would yield a, or even that a measurement of A and then B would yield a and then b. We *cannot* say that a measurement of B and then A would yield b and then a, since a measurement of B may disturb A. But weak measurements disturb less. Consider a PPS ensemble with the initial and final states $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$. A weak measurement of A on this PPS ensemble would yield a distribution peaked at a, even *after* a weak measurement of B. A weak measurement of, say, $A+B$ would yield a distribution

[2]Note that if $\langle\mathbf{\Pi}_C\rangle_w$ is real, then $\langle P\rangle$ shifts by $\langle\mathbf{\Pi}_C\rangle_w$ during the measurement. But if $\langle\mathbf{\Pi}_C\rangle_w$ is imaginary, $\langle P\rangle$ does not shift at all. Rather, it is the conjugate variable $\langle X\rangle$ that shifts: the imaginary charge shifts $\langle X\rangle$ by $i\langle\mathbf{\Pi}_C\rangle_w/\hbar\epsilon^2$ during the measurement. The conjugate variable X shifts when the initial state $\Phi(P)$ is a gaussian in P. In general, the vacuum state of the electric field is a gaussian in $\mathbf{E}$ and an imaginary weak charge could shift the vector potential $\mathbf{A}$ conjugate to the electric field $\mathbf{E}$, changing the magnetic field $\mathbf{B}$. See Prob. 16.10 and Sect. 17.5.

peaked at $a+b$, the weak value of $A+B$, even if $a+b$ is not an eigenvalue of $A+B$. This game works because weak measurements obey two rules. First, weak measurements of A and B disturb each other only slightly. Second, weak measurements are inaccurate and can yield, "by error", the weak value $\langle A+B\rangle_w = \langle A\rangle_w + \langle B\rangle_w$.

These two rules are intimately related. The second concerns the uncertainty ΔQ_d in the position of a pointer; the first concerns the conjugate momentum P_d and its uncertainty ΔP_d. A measurement of A with interaction Hamiltonian H_{int},

$$H_{int} = g(t) A P_d \ ,$$

implies a change in B that is proportional to P_d:

$$\frac{dB}{dt} = \frac{i}{\hbar} g(t) [B, A] P_d \ .$$

If the measurement is ideal, the initial wave function $\Phi_{in}(Q_d)$ of the pointer is a δ-function; then P_d and dB/dt are unbounded. Conversely if we bound dB/dt, the measurement cannot be ideal and there will be errors. For example, if we bound P_d to a finite interval, then $\Phi_{in}(Q_d)$ will be analytic and extend over the complex Q_d plane.[3] Then the expansion of the exponential in Eq. (16.13) will be valid and the measured value will be the weak value. Moreover, the shift in Q_d will not change the shape of $\Phi(Q_d)$. (See Eq. (5.8).) The game of errors is consistent.

Because the game is consistent, we can play it two different ways. We now do so and arrive at the "quantum walk". Consider a set of N spin-1/2 degrees of freedom with zero Hamiltonian. Let S_x, S_y and S_z be components of total spin, as in Eq. (16.15), and let the initial state $|\Psi_{in}\rangle$ be an eigenstate of S_x:

$$|\Psi_{in}\rangle = 2^{-N/2} \bigotimes_{i=1}^{N} (|\uparrow\rangle_i + |\downarrow\rangle_i) \ .$$

The initial wave function of the measuring device is the gaussian $\Phi_{in}(Q_d)$ in Eq. (16.2). Thus the overall initial state is $|\Psi_{in}\rangle\Phi_{in}(Q_d)$. Let the final state $|\Psi_{fin}\rangle$ of the spins be

$$|\Psi_{fin}\rangle = \bigotimes_{i=1}^{N} (\alpha_\uparrow |\uparrow\rangle_i + \alpha_\downarrow |\downarrow\rangle_i) \ ,$$

where $|\alpha_\uparrow|^2 + |\alpha_\downarrow|^2 = 1$. On the PPS ensemble with these initial and final spin states, we measure the observable $C = 2S_z/N\hbar$. Then the final state of the measuring device is, up to normalization,

$$\langle \Psi_{fin} | e^{-i2S_z P_d / N\hbar^2} | \Psi_{in}\rangle \Phi_{in}(Q_d) \ . \tag{16.20}$$

Now on the one hand, we can calculate the final state by immediately expanding the exponential in Eq. (16.20). Applying Eq. (16.13), we find that the pointer moves to the weak value

$$\langle C\rangle_w = \frac{\langle \Psi_{fin}|C|\Psi_{in}\rangle}{\langle \Psi_{fin}|\Psi_{in}\rangle} = \frac{\alpha_\uparrow - \alpha_\downarrow}{\alpha_\uparrow + \alpha_\downarrow} \ .$$

[3] For $\Phi_{in}(Q_d)$ to be analytic it is sufficient, but not necessary, to bound P_d to a finite interval. Equation (16.2) is analytic and its Fourier transform is nonzero for all P_d, vanishing exponentially in $|P_d|^2$.

Note $\langle C\rangle_w$ can be any complex number, while the eigenvalues of C range from -1 to 1. On the other hand, we can calculate the final state of the pointer by first applying $e^{-i2S_z P_d/N\hbar^2}$ to the initial state $|\Psi_{in}\rangle\Phi_{in}(Q_d)$:

$$e^{-i2S_z P_d/N\hbar^2}|\Psi_{in}\rangle\Phi_{in}(Q_d) = 2^{-N/2}\bigotimes_{i=1}^{N}\left(e^{-iP_d/N\hbar}|\uparrow\rangle_i + e^{iP_d/N\hbar}|\downarrow\rangle_i\right)\Phi_{in}(Q_d) \tag{16.21}$$

Projecting Eq. (16.21) onto the final state $|\Psi_{fin}\rangle$, we obtain the final (unnormalized) state of the pointer:

$$2^{-N/2}\left(\alpha_\uparrow e^{-iP_d/N\hbar} + \alpha_\downarrow e^{iP_d/N\hbar}\right)^N \Phi_{in}(Q_d)\ . \tag{16.22}$$

Equation (16.22) represents a superposition of many pointer states. Expanding the expression in brackets according to the binomial theorem, we see that it is a superposition of pointer states displaced by at most 1 in either direction. So how can Eq. (16.22) represent a pointer displaced by $\langle C\rangle_w$ if $\langle C\rangle_w$ is out of this range?

Apparently the pointer states interfere, *constructively* for $Q_d \approx \langle C\rangle_w$ and *destructively* for other values of Q_d. Indeed, we can verify this interference. Since

$$\begin{aligned}\alpha_\uparrow e^{-iP_d/N\hbar} + \alpha_\downarrow e^{iP_d/N\hbar} &\approx \alpha_\uparrow(1 - iP_d/N\hbar) + \alpha_\downarrow(1 + iP_d/N\hbar)\\ &= (\alpha_\uparrow + \alpha_\downarrow) - (\alpha_\uparrow - \alpha_\downarrow)iP_d/N\hbar\\ &= (\alpha_\uparrow + \alpha_\downarrow)(1 - i\langle C\rangle_w P_d/N\hbar)\end{aligned} \tag{16.23}$$

and

$$\lim_{N\to\infty}(1 - i\langle C\rangle_w P_d/N\hbar)^N = e^{-i\langle C\rangle_w P_d/\hbar}\ , \tag{16.24}$$

we find that, for large enough N, Eq. (16.22) does indeed imply the final pointer state $\Phi_{fin}(Q_d) = \Phi_{in}(Q_d - \langle C\rangle_w)$.

Mathematically speaking, Eq. (16.22) does not look like $e^{-i\langle C\rangle_w P_d/\hbar}\Phi_{in}(Q_d)$. Eq. (16.22) corresponds to a superposition of waves $e^{ikP_d/\hbar}$ where $k = -1, -1+2/N, -1+4/N, \ldots, 1$. If $e^{-i\langle C\rangle_w P_d/\hbar}$ is not one of these waves, how can we obtain it by superposing them? Surprisingly, we can: this superposition, called a *superoscillation*, is consistent with Fourier's theorem [10]. Physically, Eq. (16.22) is analogous to a random walk. We can generate a random walk in one dimension by tossing a coin at each step. In Eq. (16.22), we toss a quantum coin – a spin – to generate a quantum random walk [11]. If the coefficients $\alpha_\uparrow$ and $\alpha_\downarrow$ in Eq. (16.22) were probabilities, each term in the expansion of Eq. (16.22) would represent a random walk, with a coefficient equal to its probability. A classical random walk of N steps yields a typical displacement of $\sqrt{N}$, and never more than N. But the coefficients $\alpha_\uparrow$ and $\alpha_\downarrow$ are probability amplitudes; the quantum walk superposes classical random walks and yields arbitrary displacements.

Section 16.2 presents a paradox of two incompatible interpretations. Each interpretation is self-consistent, and the calculations to which they correspond are equivalent. We have a paradox, but no contradiction. We just have to choose an interpretation. But each interpretation

rests on an assumption. If we say that kinetic energy can be negative, we say that values measured as kinetic energy, clustering as expected for kinetic energy, around a predicted negative value of kinetic energy, *are* negative kinetic energy. This interpretation rests on the assumption that physical theories should be simple. If we say that kinetic energy cannot be negative, we say that a measured value cannot depend on what *follows* the measurement. This interpretation rests on the assumption that quantum mechanics contains an arrow of time – an assumption that Chap. 10 challenges.

16.7 Faster than Light

Negative kinetic energy and imaginary charge are just two among many surprising weak values. We now apply the quantum walk to show that the weak speed of a charged particle can exceed the speed of light *in vacuo*. Next we calculate the electromagnetic field of the particle and find that it emits Cherenkov radiation: like any charged particle moving faster than light through a medium, a superluminal particle emits Cherenkov radiation [12]. Weak superluminal speed illustrates the consistency among weak measurements on a PPS ensemble.

We start with a Hamiltonian for a particle moving at constant speed: $H = p_z v_z$, where $p_z = -i\hbar\partial/\partial z$ and v_z operates on an internal Hilbert space of the particle:

$$v_z = \frac{c}{N}\sum_{i=1}^{N}\sigma_z^{(i)} \ .$$

(The Pauli matrices represent speed, not spin; the particle has no magnetic moment.) The eigenvalues of v_z are $-c, -c + 2c/N, \ldots, c - 2c/N, c$, where c is the speed of light. The particle moves with speed v_z in the z-direction,

$$\dot{x} = [x, H]/i\hbar = 0 \ , \quad \dot{y} = [y, H]/i\hbar = 0 \ , \quad \dot{z} = [z, H]/i\hbar = v_z \ ,$$

hence the change in position z measures v_z.

If the only allowed values of v_z are its eigenvalues, the speed of the particle cannot exceed the speed of light. But consider the following weak measurement of v_z. At $t = 0$, we preselect an initial state $|\Psi_{in}\rangle\Phi(\mathbf{x}, 0)$, where $\Phi(\mathbf{x}, 0)$ represents the particle approximately localized at $\mathbf{x} = (x, y, z) = 0$,

$$\Phi(\mathbf{x}, 0) = (\epsilon^2\pi)^{-3/4} e^{-\mathbf{x}^2/2\epsilon^2} \ , \tag{16.25}$$

and $|\Psi_{in}\rangle$ is the particle's internal state. At time t we postselect the internal state $|\Psi_{fin}\rangle$. For $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$ we choose, as in the last section,

$$\begin{aligned} |\Psi_{in}\rangle &= 2^{-N/2}\bigotimes_{i=1}^{N}(|\uparrow\rangle_i + |\downarrow\rangle_i) \ , \\ |\Psi_{fin}\rangle &= \bigotimes_{i=1}^{N}(\alpha_\uparrow|\uparrow\rangle_i + \alpha_\downarrow|\downarrow\rangle_i) \ , \end{aligned} \tag{16.26}$$

with $|\alpha_\uparrow|^2+|\alpha_\downarrow|^2=1$. Our chances of postselecting the state $|\Psi_{fin}\rangle$ may be very small, but if we repeat the experiment again and again, eventually we will postselect $|\Psi_{fin}\rangle$. Then $\Phi(\mathbf{x},t)$ is

$$\Phi(\mathbf{x},t)=\langle\Psi_{fin}|e^{-ip_z v_z t/\hbar}|\Psi_{in}\rangle\Phi(\mathbf{x},0)\ , \tag{16.27}$$

up to normalization. Evaluating Eq. (16.27) exactly, we obtain

$$\Phi(\mathbf{x},t)=2^{-N/2}\left(\alpha_\uparrow e^{-ip_z ct/N\hbar}+\alpha_\downarrow e^{ip_z ct/N\hbar}\right)^N\Phi(\mathbf{x},0)\ . \tag{16.28}$$

Just as in Eqs. (16.23–24) we expand

$$\begin{aligned}\alpha_\uparrow e^{-ip_z ct/N\hbar}+\alpha_\downarrow e^{ip_z ct/N\hbar}&\approx\alpha_\uparrow(1-ip_z ct/N\hbar)+\alpha_\downarrow(1+ip_z ct/N\hbar)\\&=(\alpha_\uparrow+\alpha_\downarrow)-(\alpha_\uparrow-\alpha_\downarrow)ip_z ct/N\hbar\\&=(\alpha_\uparrow+\alpha_\downarrow)(1-ip_z\langle v_z\rangle_w t/N\hbar)\ ,\end{aligned} \tag{16.29}$$

and since

$$\lim_{N\to\infty}(1-ip_z\langle v_z\rangle_w t/N\hbar)^N=e^{-ip_z\langle v_z\rangle_w t/\hbar} \tag{16.30}$$

we find that, for large N, $\Phi(\mathbf{x},t)\approx\Phi(x,y,z-\langle v_z\rangle_w t,0)$. At time t the particle is displaced by $\langle v_z\rangle_w t$ along the z-axis. Now if $\alpha_\uparrow$ and $\alpha_\downarrow$ are real and $\alpha_\uparrow\alpha_\downarrow$ is negative, then the weak value of v_z,

$$\langle v_z\rangle_w=\frac{\langle\Psi_{fin}|v_z|\Psi_{in}\rangle}{\langle\Psi_{fin}|\Psi_{in}\rangle}=\frac{\alpha_\uparrow-\alpha_\downarrow}{\alpha_\uparrow+\alpha_\downarrow}c\ ,$$

exceeds c in magnitude. Thus the weak speed of the particle could be superluminal.

Weak superluminal speed could be an error, since the width ϵ of $\Phi(\mathbf{x},0)$ must be nonzero. (Otherwise there would be no tails in $\Phi(\mathbf{x},t)$ to interfere constructively near $z=\langle v_z\rangle_w t$.) Yet the weak value does not seem to be an error, because whenever our pre- and postselections (which are independent of $\Phi(\mathbf{x},0)$) yield $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$, respectively, measured values of the displacement of the particle over a time t cluster about $\langle v_z\rangle_w t$.

For measurements of v_z to to cluster about the weak value, the probability of postselecting $|\Psi_{fin}\rangle$ must be smaller than the probability of obtaining the weak value by error (without postselection). Otherwise, when we postselect $|\Psi_{fin}\rangle$, we would most likely not get superluminal speed. Now consider repeated weak measurements of the particle's position. Since weak measurements do not interfere with one another, the measured values will be uncorrelated; there will be no trend towards superluminal speed, even if one of the measurements shows the particle travelling with superluminal speed. Or consider the measured value $S_z=N\hbar/2$ of Sect. 16.1. The value is irrelevant to subsequent measurements of S_z, which are just as unlikely as before to yield $S_z=N\hbar/2$; it is just an error. Without postselection, there is no consistency in measurement errors. Unless and until we postselect, they are just errors. With pre- and postselection, however, measured values in this thought experiment consistently exceed c.

The charged particle has an electromagnetic field. What is the field when $\langle v_z\rangle_w$ is superluminal? Suppose first that v_z equals one of its eigenvalues, and let $V(\mathbf{x}',t;v_z)$ denote the scalar

potential at $\mathbf{x}', t$ of a particle of charge q moving along the z-axis with $z = v_z t$. We obtain $V(\mathbf{x}', t; v_z)$ via a Lorentz boost, by v_z in the z-direction, of the Coulomb potential $V(\mathbf{x}', t; 0)$:

$$V(\mathbf{x}', t; v_z) = q\left\{[(x')^2 + (y')^2](1 - v_z^2/c^2) + (z' - v_z t)^2\right\}^{-1/2} \ . \tag{16.31}$$

We obtain the vector potential $\mathbf{A}(\mathbf{x}', t; v_z)$ in the same way; it has only one nonzero component,

$$A_z(\mathbf{x}', t; v_z) = \frac{v_z}{c} V(\mathbf{x}', t; v_z) \ .$$

These are the classical potentials of a point charge moving along the axis with $z = v_z t$. But our moving charge is a quantum particle, and its potentials are quantum potentials. To treat these potentials, we simply postulate an effective two-particle interaction between the moving charge and a test particle. That is, to the Hamiltonian $H = p_z v_z$ of the moving charge we add the Hamiltonian H' of a (nonrelativistic) test particle:

$$H' = \frac{1}{2m}\left(\mathbf{p}' - q'\mathbf{A}\right)^2 + q'V \ . \tag{16.32}$$

In H', the test particle has mass m and charge q', and the potentials are

$$\begin{aligned} V(\mathbf{x}') &= q\left\{\left[(x' - x)^2 + (y' - y)^2\right](1 - v_z^2/c^2) + (z' - z)^2\right\}^{-1/2} \ , \\ A_z(\mathbf{x}') &= \frac{q v_z}{c}\left\{\left[(x' - x)^2 + (y' - y)^2\right](1 - v_z^2/c^2) + (z' - z)^2\right\}^{-1/2} \ . \end{aligned} \tag{16.33}$$

The equations of motion obtained from $H + H'$ yield $(x, y, z) = (0, 0, v_z t)$ together with the correct motion of the test particle in the electromagnetic field of the moving charge.[4] Now we treat V and A_z as quantum operators and calculate their effect on the test particle. We will see that if the moving charge has weak speed $\langle v_z \rangle_w$, then $\langle v_z \rangle_w$ takes the place of v_z in Eqs. (16.33).

We preselect the state $|\Psi_{in}\rangle\Phi(\mathbf{x}, 0)$ of the moving charge and, after a time T, postselect the state $|\Psi_{fin}\rangle$. (See Eqs. (16.25–26).) We also prepare the test particle in a localized state $\Omega(\mathbf{x}', 0)$, where $\Omega(\mathbf{x}', 0)$ is analytic in $\mathbf{x}'$. Since we want the test particle to measure the instantaneous values of V and A_z at the end of this evolution (and not their average values during or after the evolution), we "turn on" H' instantaneously at time T. (Formally, we multiply H' by $\delta(t - T)$.) The state of the moving charge and the test particle after the postselection is then

$$\Phi(\mathbf{x}, T)\Omega(\mathbf{x}', T) = \langle \Psi_{fin}| e^{-i[(\mathbf{p}' - q'\mathbf{A})^2/2m + q'V]/\hbar} e^{-ip_z v_z T/\hbar} |\Psi_{in}\rangle \Phi(\mathbf{x}, 0)\Omega(\mathbf{x}', 0) \ . \tag{16.34}$$

The potentials V and $\mathbf{A}$ in Eq. (16.34), as defined by Eqs. (16.33), contain v_z. But we can show that the weak speed $\langle v_z \rangle_w$ should replace v_z in Eq. (16.34). We begin with the right-hand side of Eq. (16.34). We can expand the first exponential,

$$e^{-i[(\mathbf{p}' - q'\mathbf{A})^2/2m + q'V]/\hbar} \ ,$$

[4] The equation of motion $(x, y, z) = (0, 0, v_z t)$ implies $V(\mathbf{x}') = V(\mathbf{x}', t; v_z)$ and $A_z(\mathbf{x}') = A_z(\mathbf{x}', t; v_z)$. The equation of motion for the momentum $\mathbf{p}$ of the moving charge is unphysical, but it has no measurable consequences.

as a power series in v_z. Thus, the right-hand side of Eq. (16.34) is a sum of terms of the form

$$\langle \Psi_{fin}| v_z^n e^{-ip_z v_z T/\hbar} |\Psi_{in}\rangle$$

multiplied on either side by functions that do not depend on v_z. Now we have, for any n and in the limit $N \to \infty$,

$$\begin{aligned}\langle \Psi_{fin}| v_z^n e^{-ip_z v_z T/\hbar} |\Psi_{in}\rangle &= \left(\frac{i\hbar}{T}\frac{\partial}{\partial p_z}\right)^n \langle \Psi_{fin}| e^{-ip_z v_z T/\hbar} |\Psi_{in}\rangle \\ &= \langle \Psi_{fin}|\Psi_{in}\rangle \left(\frac{i\hbar}{T}\frac{\partial}{\partial p_z}\right)^n e^{-ip_z \langle v_z\rangle_w T/\hbar} \\ &= \langle \Psi_{fin}|\Psi_{in}\rangle (\langle v_z\rangle_w)^n e^{-ip_z \langle v_z\rangle_w T/\hbar} \ . \end{aligned} \tag{16.35}$$

(Compare Eqs. (16.27–30).) That is, we can simply replace v_z by $\langle v_z\rangle_w$ everywhere it appears in the power series. We drop the factor $\langle \Psi_{fin}|\Psi_{in}\rangle$ (to normalize) and obtain

$$\Phi(\mathbf{x}, T)\Omega(\mathbf{x}', T) = e^{-i[(\mathbf{p}' - q'\mathbf{A})^2/2m + q'V]/\hbar} \Phi(x, y, z - \langle v_z\rangle_w T, 0)\Omega(\mathbf{x}', 0) \ ,$$

where

$$A_z = \frac{\langle v_z\rangle_w}{c} V = \frac{\langle v_z\rangle_w}{c} V(\mathbf{x}' - \mathbf{x}, 0; \langle v_z\rangle_w) \ .$$

Since $V(\mathbf{x}' - \mathbf{x}, 0; \langle v_z\rangle_w)$ equals $V(\mathbf{x}')$ as defined in Eqs. (16.33) with $\langle v_z\rangle_w$ taking the place of v_z, the scalar and vector potentials are exactly the potentials of a charge moving with weak speed $\langle v_z\rangle_w$ (folded with the width of the localized state Φ) and have the corresponding effect on the test particle. Now if $\langle v_z\rangle_w$ exceeds the speed of light, V and A_z correspond to Cherenkov radiation, the shock wave of a charged particle moving faster than light through a medium.[5]

16.8 Galilean Dialogue

[Simplicio, Salviati and Sagredo stop at the Kinneret beach.]

SAGREDO. Frankly, Salviati, I again feel the sand slipping through my fingers. You want *two* states – a past state evolving forwards in time and a future state evolving backwards in time – to describe a quantum system. Negative kinetic energy is remarkable – even irresistable – but how far can we control future states? The world we live in does contain an arrow of time, even if quantum theory does not, so these measurements on pre- and post-selected ensembles are artificial.

SIMPLICIO. As for me, I was unable to follow all the mathematics, and it is still quite unclear to me when to call a measured value an error and when to call it a weak value. But I was struck by your remark that a measured negative value of kinetic energy – even if it exactly coincides with the predicted negative value – is not yet a measured *weak* value, unless and until we postselect.

[5] For $\langle v_z\rangle_w$ superluminal, $V(\mathbf{x}' - \mathbf{x}, 0, \langle v_z\rangle_w)$ vanishes outside a cone trailing the source. For imaginary weak speed see Sects. 16.5 and 17.5.

SALVIATI. Otherwise we would deny the very possibility of errors in measurements, which of course we do not. The correlation between weak measurements and postselection is – must be – a one-way correlation. Postselecting $|\Psi_{fin}\rangle$ after weak measurements of A on $|\Psi_{in}\rangle$, we find that the measured values cluster around $\langle A\rangle_w$, however strange $\langle A\rangle_w$ may be. But unless we postselect $|\Psi_{fin}\rangle$, the weak measurements rarely yield $\langle A\rangle_w$, and those that do, do not improve our chances of postselecting $|\Psi_{fin}\rangle$. Therefore, the probability that the measurement yields $\langle A\rangle_w$ by error, small as it is, must exceed the probability of postselecting $|\Psi_{fin}\rangle$.

SAGR. That is what bothers me – the miniscule probability of postselecting $|\Psi_{fin}\rangle$ and the conditions on $|\Psi_{fin}\rangle$.

SALV. Sagredo, the examples may seem artificial, but pre- and postselected states describe the quantum world more completely than do preselected states alone. Shouldn't a more complete description come at higher price? Small probabilities and conditions on the postselected state are the price we pay for a more complete description.

Postselection turns measurement errors into superluminal speed and fluctuations of the vacuum into Cherenkov radiation. The description is complete and consistent: a measured weak value may strike us as madness, but then we look at other weak values measured on the same PPS ensemble and see method in the madness.

SIMP. Superluminal speed is not consistent with relativistic causality!

SAGR. Oh, but it is, Simplicio. We assumed that $\Phi(\mathbf{x}, 0)$, the wave function of the moving charge, is analytic, just as the wave function $\Phi_{in}(Q_d)$ of a pointer must always be analytic. The pointer shows the weak value only if $\Phi_{in}(Q_d)$ is analytic, because then $\Phi_{in}(Q_d)$ does not vanish for any interval in Q_d and can show, "by error", the weak value. Likewise $\Phi(\mathbf{x}, 0)$ is analytic in z, so $\Phi(\mathbf{x}, 0)$ and its derivatives at a given z determine $\Phi(\mathbf{x}, 0)$ for all z. Hence $\Phi(\mathbf{x}, t) = \Phi(x, y, z - \langle v_z\rangle_w t, 0)$ does not transmit any message, for the message is the same for all z and t. Since it does not transmit any message, it does not transmit any superluminal message, and weak superluminal speed is consistent with relativistic causality.

SIMP. Where is free will in this "complete description"?

SALV. Weak values are compatible with free will! Following a measurement of A we can either postselect, or measure A again. If we postselect, we may interpret the result of the measurement of A as a weak value $\langle A\rangle_w$; if we remeasure A, we may interpret the same result as an error. These two interpretations are consistent, for they apply to different ensembles – the former to a pre- and postselected ensemble and the latter to a preselected ensemble. How we interpret a measured value depends on what else we choose to measure, but no one tells us what to measure.

Problems

16.1 (a) A particle tunnels through a one-dimensional square well potential; its wave function is a superposition of eigenstates of the Hamiltonian with energies below the potential barrier. Show that the wave function does not vanish outside the potential barrier. (b) For any wave function that vanishes outside the potential barrier, show that the expectation value of the total energy is always greater than the potential barrier.

16.2 Let $V(x)$ in Eq. (16.3) equal $V(x) = -(\hbar^2\alpha/m)\delta(x)$. Show that H has a single bound state $\Psi_{in}(x) = \sqrt{\alpha}e^{-\alpha|x|}$ with energy $-\alpha^2\hbar^2/2m$.

16.3 Derive Eq. (16.10) including normalization, and show that the integral converges.

*16.4 (a) Show that an upper bound on Eq. (16.10) is

$$\frac{\hbar\alpha}{\pi}\frac{e^{(\alpha-p_0/\hbar)x_0-\alpha^2\delta^2/2}}{p_0^2-\alpha^2\hbar^2}\int dp\ e^{-(p^2-p_0^2)\delta^2/2\hbar^2}|\Phi_{in}\left(Q_d-(p-ip_0)^2/2m\right)|\ .$$

(b) Show that this upper bound leads to the exponential e^{Σ} where Σ is

$$\Sigma=(\alpha-p_0/\hbar)x_0-\delta^2\left(\frac{p_0^2}{\hbar^2}+\frac{\alpha^2}{2}\right)+Q_d\left(\frac{p_0^2}{m\epsilon^2}-\frac{\delta^2 m}{\hbar^2}\right)+\frac{p_0^4}{m^2\epsilon^2}+\frac{\delta^4m^2\epsilon^2}{2\hbar^4}\ .$$

(c) Eliminate the dependence on Q_d in Σ by setting $p_0 = \delta m\epsilon/\hbar$. Since $p_0 > \alpha\hbar$ we then obtain Eq. (16.5). Now Eq. (16.10), the correction to Eq. (16.8), is negligible if $\Sigma \ll 0$. Derive Eq. (16.6) from the condition $\Sigma \ll 0$.

16.5 For an oscillator of mass m and angular momentum ω, calculate the weak value $\langle m\omega x^2/\hbar + p^2/m\omega\hbar\rangle_w$,

$$\langle m\omega x^2/\hbar+p^2/m\omega\hbar\rangle_w=\frac{\langle\Psi_{fin}|(m\omega x^2/\hbar+p^2/m\omega\hbar)|\Psi_{in}\rangle}{\langle\Psi_{fin}|\Psi_{in}\rangle}\ ,$$

assuming

$$[m\omega(x+x_0)^2/\hbar+(p+p_0)^2/m\omega\hbar]|\Psi_{in}\rangle=|\Psi_{in}\rangle$$

and

$$[m\omega(x-x_0)^2/\hbar+(p-p_0)^2/m\omega\hbar]|\Psi_{fin}\rangle=|\Psi_{fin}\rangle\ .$$

*16.6 Consider the following condition [13] on an observable A:

$$\frac{2^{n/2}\Gamma(n/2)}{\epsilon^n(n-2)!}\left[\langle A^n\rangle_w-\langle A\rangle_w^n\right]\ll 1\ .$$

Show that if ϵ in Eq. (16.2) satisfies this condition for $n = 2, 3, \ldots$, then we can neglect the corrections to Eq. (16.13).

16.7 Show that the measurement of A in Eq. (16.16) yields $\langle A\rangle_w$ in the limit of large N, for any given ϵ.

16.8 (a) If a measurement of A on either $|\Psi_{in}\rangle$ or $|\Psi_{fin}\rangle$ yields a with certainty, show that

$$\langle A\rangle_w=\frac{\langle\Psi_{fin}|A|\Psi_{in}\rangle}{\langle\Psi_{fin}|\Psi_{in}\rangle}=a\ .$$

(b) Show that the converse is true, in the special case that A is a nondegenerate observable on a two-dimensional Hilbert space: if the weak value $\langle A\rangle_w$ equals an eigenvalue a of A, then either $|\Psi_{in}\rangle$ or $|\Psi_{fin}\rangle$ is the corresponding eigenstate.

16.9 Consider a system with states $|1\rangle$ and $|2\rangle$; its Hamiltonian is $H = E_1\mathbf{\Pi}_1 + E_2\mathbf{\Pi}_2$, where $\mathbf{\Pi}_i = |i\rangle\langle i|$. We couple the system to a single spin. The interaction Hamiltonian for the system and spin is $H_{int} = \alpha\mathbf{\Pi}_2\sigma_z$, and $H + H_{int}$ is the complete Hamiltonian for the coupled system and spin. At time $t = 0$, the state of the system and spin is $(|1\rangle + |2\rangle) \otimes (|\uparrow\rangle + i|\downarrow\rangle)/2$.
(a) Calculate the exact state of the system at time $t = T$. At time $t = T$ we postselect the spin in the state $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. Show that, for small enough α, the Hamiltonian for the system includes an effective nonunitary part $-i\alpha\mathbf{\Pi}_2$.
(b) Use the ABL formula, Eq. (10.8), to show that the probablility of finding the system in the state $|2\rangle$ at times $0 \le t < T$ is half, independent of t. Hence the Hamiltonian of the system is effectively nonunitary only *after* the postselection.

16.10 (a) Consider a weak measurement of $\mathbf{\Pi}_C$ according to H_{int} in Eq. (16.18), and let the initial state of the measuring device be $\Phi(P)$ in Eq. (16.19). Show that if $\langle\mathbf{\Pi}_C\rangle_w$ is real, then $\langle P\rangle$ shifts from $\langle P\rangle = 0$ to $\langle P\rangle = \langle\mathbf{\Pi}_C\rangle_w$ during the measurement, while if $\langle\mathbf{\Pi}_C\rangle_w$ is imaginary, $\langle P\rangle$ does not shift at all; rather, it is $\langle X\rangle$ that shifts from $\langle X\rangle = 0$ to $\langle X\rangle = i\langle\mathbf{\Pi}_C\rangle_w/\hbar\epsilon^2$.
(b) Now let the initial state of measuring device be

$$\Phi(P) = \frac{\sin\epsilon P}{\sqrt{\pi}P}$$

and calculate the final state of the measuring device after the weak measurement of $\mathbf{\Pi}_C$.

16.11 Prove that if the Fourier transform $\tilde{f}(P)$ of a function $f(Q)$ vanishes outside a finite interval in P, then $f(Q)$ is analytic.

*16.12 Define

$$I(n, z, t) = \int_{-\pi}^{\pi} d\theta e^{in\theta - i(n/z)\sin\theta} e^{it\cos\theta} ,$$

where n is an integer and z and t are real. Note that for any n and z, $I(n, z, t)$ is a sum of waves $e^{i\omega t}$ where $|\omega| \le 1$. But prove that

$$\lim_{n\to\infty} I(n, z, t)/I(n, z, 0) = e^{izt}$$

for every $z > 1$.

*16.13 (a) For μ a constant and N positive, derive the following Taylor series:

$$\left(1 + \frac{\mu}{N}\right)^N = e^{\mu}\left(1 - \frac{\mu^2}{2N} + \frac{3\mu^4 + 8\mu^3}{24N^2} + \dots\right) .$$

(b) Eqs. (16.27–30) imply that $\Phi(\mathbf{x}, t) = \Phi(x, y, z - \langle v_z\rangle_w t, 0)$ in the limit $N \to \infty$. What if we do not take the limit $N \to \infty$? Apply the Taylor series to show that

$$N\epsilon^2 \gg \langle v_z\rangle_w^2 t^2$$

is a sufficient condition for a weak measurement of $\Phi(\mathbf{x}, t) \approx \Phi(x, y, z - \langle v_z\rangle_w t, 0)$.

16.14 Given a system in an initial state $|\Psi_{in}\rangle$, the probability of postselecting $|\Psi_{fin}\rangle$ is $p \equiv |\langle\Psi_{fin}|\Psi_{in}\rangle|^2$. *Before* the postselection, suppose that a weak measurement of A on $|\Psi_{in}\rangle$ yields

$$\langle A\rangle_w = \frac{\langle\Psi_{fin}|A|\Psi_{in}\rangle}{\langle\Psi_{fin}|\Psi_{in}\rangle} ,$$

with $\langle A\rangle_w$ real. Is the probability of postselecting the state $|\Psi_{fin}\rangle$ now larger than p? Consider a fast measurement sequence (during which we can neglect the Hamiltonian of the system): first, preparation of systems in the state $|\Psi_{in}\rangle$ and of measuring devices in the state $\Phi_{in}(Q_d) = (\epsilon^2\pi)^{-1/4}e^{-Q_d^2/2\epsilon^2}$; second, measurements of A that leave the measuring devices and measured systems in the state

$$e^{-iAP_d/\hbar}|\Psi_{in}\rangle\Phi_{in}(Q_d) ;$$

third, postselection of the state $\Phi_{in}(Q_d - \langle A\rangle_w)$ of the *measuring devices*. Show that, to order $1/\epsilon$, the probability of postselecting $|\Psi_{fin}\rangle$ after this sequence is p; to order $1/\epsilon^3$, it differs from p by

$$\frac{p}{2\epsilon^2}[2\langle A\rangle_w^2 - 2\langle A\rangle\langle A\rangle_w + \langle A^2\rangle - \langle A^2\rangle_w]$$

where $\langle A\rangle$ and $\langle A^2\rangle$ are expectation values in the state $|\Psi_{in}\rangle$.

*16.15 (a) A particle with charge q produces a Coulomb field of strength $E = q/D^2$ at a distance D. If the particle sits on the z-axis, show that a measurement of its Coulomb field at a distance D yields its position along the z-axis with uncertainty $\Delta z = (D^3/2q)\Delta E$.
(b) Consider the thought experiment of Sect. 16.7. The width of the wave function $\Phi(\mathbf{x}, 0)$ of Eq. (16.25) must satisfy the condition $N\epsilon^2 \gg \langle v_z\rangle_w^2 t^2$ of Prob. 16.13 for a weak measurement. But let an observer a distance D from the charge measure its position at $t = 0$ with uncertainty $\Delta z \ll \epsilon$. Then the width of $\Phi(\mathbf{x}, 0)$ will not satisfy the condition. If $D \leq ct$ the observer may have causally disturbed the charge, but if $D > ct$ there cannot be any causal influence of the observer on the charge. So do we conclude that the measurement cannot be weak? Derive $\sqrt{N}D^2\Delta E \gg 2q$ as the condition for a weak measurement in this case, assuming $\langle v_z\rangle_w > c$.
(c) Now apply Prob. 8.14 to obtain $\hbar c > 4q^2/N$ as a sufficient condition for a weak measurement.

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17 Weak Values and Entanglement

Chapter 1 declares that we will not stop asking about quantum mechanics, "But how can it be like that?" Here and there, the declaration is hard to uphold, and nowhere harder than in Chap. 3. Chapter 3 sets out the only reasonable explanation of quantum entanglement – and shows that it doesn't work; as Bell put it, "The reasonable thing just doesn't work." Still we ask, "But how can entanglement be like that?"

In this chapter, we study entanglement using weak values. We consider weak measurements between a preselected state $|\Psi_{in}\rangle$ and a postselected state $|\Psi_{fin}\rangle$, where $|\Psi_{in}\rangle$ is entangled. For example, the paradox in the next section involves an electron and a positron between pre- and postselected states, where the preselected state is entangled. Weak measurements show these particles to be entangled in unexpected ways.

17.1 Interaction-free Paradox

Section 6.4 describes the interaction-free measurement of Elitzur and Vaidman. The measuring device is a Mach-Zehnder interferometer. (See Fig. 6.8.) If both paths through the interferometer are clear, a photon passing through it exits in the direction of constructive interference. But assume that a "bomb" (a detector) blocks one of the paths, and the bomb is certain to explode if a photon hits it (the detector is perfectly efficient). Even so, the bomb may not explode, and the photon may exit the interferometer in the direction of destructive interference. The bomb, by not exploding, reveals which path the photon took, and thus eliminates the interference between the two paths. Or, as Sect. 6.4 puts it, the bomb acts at a distance on the photon by changing the expectation value of a modular operator.

Consider now sending electrons or positrons, instead of photons, through such an interferometer. Hardy [1] invented a paradox with two interferometers, one for electrons and the other for positrons. The interferometers overlap in a corner where a path in one interferometer crosses a path in the other. (See Fig. 17.1.) The momenta of the particles and the dimensions of the interferometers are such that electrons passing through the electron interferometer always arrive at detector C_- if no positron enters their interferometer, and positrons passing through the positron interferometer always arrive at detector C_+ if no electron enters their interferometer. (The C in C_- and C_+ denotes constructive interference.) But if an electron and a positron pass through the interferometers simultaneously, they may annihilate each other in the overlapping corner. Let us assume that an electron and a positron passing through simultaneously always annihilate each other whenever their paths cross. If the particles annihilate each other, no detector clicks. What if they do not annihilate each other?

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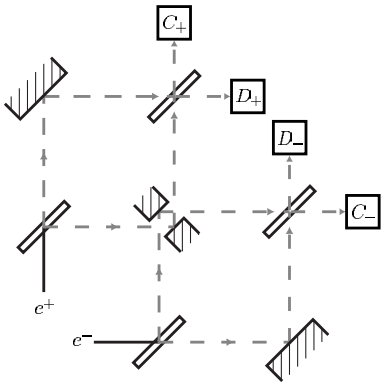

Figure 17.1: Two Mach-Zehnder interferometers with overlapping paths.

If the paths of the particles do not cross, there are three possibilities. The positron may take its overlapping path while the electron takes its nonoverlapping path. We will denote this state $|o\rangle_+ \otimes |no\rangle_-$. Or the electron may take its overlapping path while the positron takes its nonoverlapping path; then their state is $|no\rangle_+ \otimes |o\rangle_-$. Finally, both particles may take nonoverlapping paths. Then the state is $|no\rangle_+ \otimes |no\rangle_-$. All three states are equally probable, and if we absorb their phases into the definitions of the states $|o\rangle_+$, $|o\rangle_-$, $|no\rangle_+$ and $|no\rangle_-$, the state of the particles as they exit the interferometers (before they arrive at the detectors) is

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{3}} \left[|o\rangle_+ \otimes |no\rangle_- + |no\rangle_+ \otimes |o\rangle_- + |no\rangle_+ \otimes |no\rangle_- \right] .$$

If we project $|\Psi_{in}\rangle$ onto the positron's nonoverlapping path $|no\rangle_+$, we leave the electron in the state ${}_+\langle no|\Psi_{in}\rangle = [|o\rangle_- + |no\rangle_-]/\sqrt{2}$. We have assumed that in this case detector C_- clicks. Hence detector D_- must click for the orthogonal state $[|o\rangle_- - |no\rangle_-]/\sqrt{2}$. Similarly, detector C_+ clicks for the positron state ${}_-\langle no|\Psi_{in}\rangle = [|o\rangle_+ + |no\rangle_+]/\sqrt{2}$ and detector D_+ clicks for the orthogonal state $[|o\rangle_+ - |no\rangle_+]/\sqrt{2}$. Now if we compute joint probabilities for electrons and positrons, we find that detectors C_- and C_+ click together with probability 3/4, and each other pair of detectors (C_- and D_+, D_- and C_+, D_- and D_+) clicks with probability 1/12, if the particles do not annihilate each other.

What do these clicks tell us? If the detector D_+ clicks, it tells us that the electron crossed into the positron interferometer; for, by assumption, positrons always arrive at detector C_+ if no electron enters their interferometer. Thus we conclude that the electron took its overlapping path (and the positron took its nonoverlapping path). Similarly, if the detector D_- clicks, it tells us that the positron crossed into the electron interferometer; for, by assumption, electrons always arrive at detector C_- if no positron enters their interferometer. Thus we conclude that the positron took its overlapping path (and the electron took its nonoverlapping path).

But what if detectors D_- and D_+ click together? We may conclude from the clicking of detector D_+ that the electron took its overlapping path, and we may conclude from the clicking of detector D_- that the positron took *its* overlapping path. But if *both* particles took their overlapping paths, they must have annihilated each other! The simultaneous clicking of D_- and D_+ leads to the paradoxical conclusion that neither D_- nor D_+ could have clicked.

Thus the logic that explains the Elitzur-Vaidman experiment does not explain Hardy's experiment; it produces a paradox, not an explanation. We can describe the electron and the positron via modular variables, but they do not resolve the paradox. (See Prob. 17.1.)

17.2 A Grin Without a Cat

Cats come and go, but not as the grinning Cheshire Cat would come and go:

> "Did you say 'pig,' or 'fig'?" said the Cat.
>
> "I said 'pig,' " replied Alice; "and I wish you wouldn't keep appearing and vanishing so suddenly: you make one quite giddy!"
>
> "All right," said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone.
>
> "Well! I've often seen a cat without a grin," thought Alice; "but a grin without a cat! It's the most curious thing I ever saw in all my life!" [2]

Indeed, it *is* curious; but like many curious things, it has a quantum explanation. This section explains how the Cheshire Cat could disappear while leaving its grin behind.

Cats are complicated systems. (See Chap. 9.) To simplify, we approximate the Cheshire Cat as a particle with two grin states, $|\uparrow\rangle$ and $|\downarrow\rangle$; in the state $|\uparrow\rangle$ the Cat grins. (In the state $|\downarrow\rangle$ it frowns.) Two states, $|1\rangle$ and $|2\rangle$, represent locations of the Cat in two cozy boxes. We assume that the Hamiltonian of the Cat commutes with the projection operators for the grin state

$$\mathbf{\Pi}_\uparrow \equiv |\uparrow\rangle\langle\uparrow| \,, \quad \mathbf{\Pi}_\downarrow \equiv |\downarrow\rangle\langle\downarrow|$$

and the location

$$\mathbf{\Pi_1} \equiv |1\rangle\langle 1| \,, \quad \mathbf{\Pi_2} \equiv |2\rangle\langle 2|$$

of the Cat. At time $t = 0$, we preselect the Cat in the state

$$|\Psi_{in}\rangle = \frac{1}{2}\left[|\uparrow\rangle + |\downarrow\rangle\right]|1\rangle \; + \; \frac{1}{2}\left[|\uparrow\rangle - |\downarrow\rangle\right]|2\rangle \,,$$

and we postselect it at time $t = T$ in the state

$$|\Psi_{fin}\rangle = \frac{1}{2}[|\uparrow\rangle - |\downarrow\rangle]\,[|1\rangle + |2\rangle] \;.$$

Where is the Cat at intermediate times? To locate the Cat, we calculate the weak values of $\mathbf{\Pi_1}$ and $\mathbf{\Pi_2}$. At times $0 \le t \le T$ we have $\langle \mathbf{\Pi_1} \rangle_w = 0$ and $\langle \mathbf{\Pi_2} \rangle_w = 1$. So, clearly, the Cat passes its time in box 2. Is it grinning during this time? It sure is! We calculate that $\langle \sigma_z \rangle_w = \langle \mathbf{\Pi}_\uparrow - \mathbf{\Pi}_\downarrow \rangle_w = 1$, so at all intermediate times, the Cat is grinning.

The Cat is grinning, and the Cat is in box 2. Hence it is obvious, dear Alice, that the Cat is grinning in box 2. Obvious, but false: the weak value of $\sigma_z \mathbf{\Pi_2}$ *vanishes*. There is no net

grin in box 2! We can measure the Cat's grin and location simultaneously (since σ_z commutes with the projectors $\mathbf{\Pi_1}$ and $\mathbf{\Pi_2}$); but the weak value of the product $\langle \sigma_z \mathbf{\Pi_i} \rangle_w$ does not equal the product of the weak values $\langle \sigma_z \rangle_w$ and $\langle \mathbf{\Pi_i} \rangle_w$. Thus, although the Cat grins and the Cat is in box 2, we cannot conclude that the grin is in box 2! On the contrary, the grin is in box 1 – the weak value of $\sigma_z \mathbf{\Pi_1}$ is 1. The Cheshire Cat is in box 2 but its grin is in box 1.

A grin without a cat! Although the preselected state $|\Psi_{in}\rangle$ entangles the location of the Cat and its grin state, we seem to have quite *dis*entangled the grin and the Cat. To analyze this effect, let us calculate the weak values of products of the projectors $\mathbf{\Pi_1}$, $\mathbf{\Pi_2}$, $\mathbf{\Pi}_\uparrow$ and $\mathbf{\Pi}_\downarrow$. We have

$$\begin{aligned}
\langle \mathbf{\Pi_1 \Pi}_\uparrow \rangle_w &= 1/2 \,, \\
\langle \mathbf{\Pi_1 \Pi}_\downarrow \rangle_w &= -1/2 \,, \\
\langle \mathbf{\Pi_2 \Pi}_\uparrow \rangle_w &= 1/2 \,, \\
\langle \mathbf{\Pi_2 \Pi}_\downarrow \rangle_w &= 1/2 \,.
\end{aligned} \tag{17.1}$$

These four weak values add up to 1, as they must since $\mathbf{\Pi}_\uparrow + \mathbf{\Pi}_\downarrow = 1 = \mathbf{\Pi_1} + \mathbf{\Pi_2}$. The four products $\mathbf{\Pi_1 \Pi}_\uparrow$, $\mathbf{\Pi_1 \Pi}_\downarrow$, $\mathbf{\Pi_2 \Pi}_\uparrow$ and $\mathbf{\Pi_2 \Pi}_\downarrow$ sum to 1 and ordinarily represent occupation; e.g. $\mathbf{\Pi_1 \Pi}_\uparrow$ represents the occupation of box 1 by the grinning Cat. But while the eigenvalues of a projector can be only 0 or 1, and the expectation value of a projector in any state must lie in the interval $[0, 1]$, the weak value of $\mathbf{\Pi_1 \Pi}_\downarrow$ is *negative*.

"Curiouser and curiouser!" as Alice might say. Yet if we interpret these weak values carefully, they give a consistent account of the Cheshire Cat. The weak location of the Cat is in box 2, because $\langle \mathbf{\Pi_2} \rangle_w$ is the sum of $\langle \mathbf{\Pi_2 \Pi}_\uparrow \rangle_w$ and $\langle \mathbf{\Pi_2 \Pi}_\downarrow \rangle_w$, both of which equal $1/2$; it could not be in box 1, because the sum of $\langle \mathbf{\Pi_1 \Pi}_\uparrow \rangle_w$ and $\langle \mathbf{\Pi_1 \Pi}_\downarrow \rangle_w$ vanishes. The net weak grin in box 2 vanishes, because $\langle \mathbf{\Pi_2 \Pi}_\uparrow \rangle_w$ represents a grinning Cat and $\langle \mathbf{\Pi_2 \Pi}_\downarrow \rangle_w$ represents a frowning Cat. But a frown is an inverted grin, i.e. it is a grin with respect to an inverted axis. (See Fig. 17.2.) Since the weak values of the grin and the frown in box 2 are equal, they cancel one another. Most curious is the grin in box 1. The weak value of the frown in box 1 is

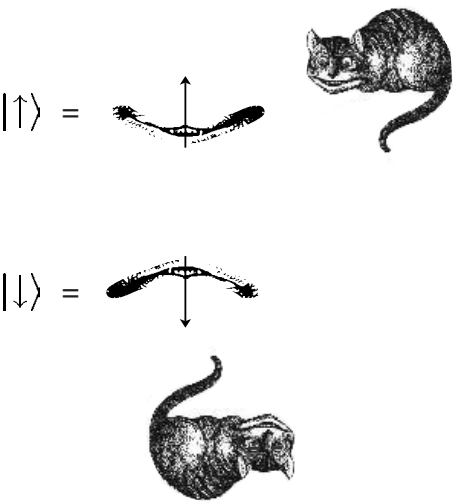

Figure 17.2: Cheshire Cat grin states. A frown is a grin with respect to the inverted axis.

$-1/2$; but a negative weak value for a frown counts as a positive weak value for a grin, hence $\langle \mathbf{\Pi_1 \Pi_\uparrow} \rangle_w$ and $\langle \mathbf{\Pi_1 \Pi_\downarrow} \rangle_w$ combine to yield a grin without a cat.

As noted, cats are complicated systems. We cannot yet separate a cat from its grin, but we could try separating a simpler system from its internal degrees of freedom. For example, we could try separating a neutron from its magnetic moment; the equations of this section apply to a neutron and its magnetic moment just as they apply to the Cat and its grin. We preselect neutrons in a neutron interferometer [3] in the state $|\Psi_{in}\rangle$ and postselect them in the state $|\Psi_{fin}\rangle$, where now $|\uparrow\rangle$ and $|\downarrow\rangle$ refer to neutron spin states and $|1\rangle$ and $|2\rangle$ refer to the two arms of the interferometer. Any measurement, weak or strong, of the neutron's location will find the neutron in state $|2\rangle$, i.e. will locate the neutron in arm 2 of the interferometer. Any measurement will also show the neutron to have $\sigma_z = 1$. (See Prob. 17.2.) A *weak* measurement of $\sigma_z \mathbf{\Pi_2}$ will show that $\langle \sigma_z \mathbf{\Pi_2} \rangle_w$ vanishes, while a weak measurement of $\sigma_z \mathbf{\Pi_1}$ will show that $\langle \sigma_z \mathbf{\Pi_1} \rangle_w$ equals 1. What does it mean to measure $\sigma_z \mathbf{\Pi_1}$ and $\sigma_z \mathbf{\Pi_2}$ weakly? We could, for example, weakly measure the magnetic field in both arms of the interferometer; the measurements would show that the magnetic moment of the neutron takes arm 1 of the interferometer. The magnetic moment takes one arm of the interferometer but the neutron itself takes the other arm!

17.3 Alice and Bob in Wonderland

Let us apply weak values to the EPR-Bohm experiment. (See Sect. 3.4.) Alice and Bob share many pairs of spin-1/2 systems prepared in an initial singlet state

$$|\Psi_{in}\rangle = [|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B]/\sqrt{2} \, .$$

Alice measures the spin component of each of her systems along an axis that she chooses at random from a finite list of axes; Bob does likewise for his systems, from the same list. For example, on some pairs Alice may measure the spin component along the x-axis while Bob measures the spin component along the y-axis. Let us assume that they do, and we consider only those pairs for which $\sigma_x^A = 1$ and $\sigma_y^B = 1$; that is, we postselect the state

$$|\Psi_{fin}\rangle = [|\uparrow\rangle_A + |\downarrow\rangle_A] \otimes [|\uparrow\rangle_B + i|\downarrow\rangle_B]/2 \, . \tag{17.2}$$

Here is a formulation of the EPR paradox: we assume, with EPR, that the results of Alice's measurements cannot depend on Bob's choice of axis, and vice versa. Hence the systems, before reaching Alice's and Bob's respective laboratories, must carry (at least) definite values of σ_x^A, σ_y^A, σ_x^B and σ_y^B, all independent of what axes Alice and Bob will choose. Now note that $|\Psi_{in}\rangle$ is an eigenvalue of the three operators $\sigma_x^A \sigma_x^B$, $\sigma_y^A \sigma_y^B$ and $\sigma_x^A \sigma_y^B + \sigma_y^A \sigma_x^B$:

$$\sigma_x^A \sigma_x^B |\Psi_{in}\rangle = \sigma_y^A \sigma_y^B |\Psi_{in}\rangle = -|\Psi_{in}\rangle \tag{17.3}$$

and

$$\left(\sigma_x^A \sigma_y^B + \sigma_y^A \sigma_x^B\right) |\Psi_{in}\rangle = 0 \, . \tag{17.4}$$

From these eigenvalue equations we can draw conclusions about observables that Alice and Bob did *not* measure on these postselected pairs. From Eq. (17.3) we conclude that $\sigma_x^B = \sigma_y^A = -1$, while from Eq. (17.4) we conclude that $\sigma_y^A \sigma_x^B = -1$. But these are incompatible conclusions!

How can it be like that? Let us proceed, as we did in the last section, by calculating weak values of projectors. We define four projectors, $\mathbf{\Pi}^A_{\uparrow y}$ and $\mathbf{\Pi}^A_{\downarrow y}$ projecting onto states with $\sigma_y^A = \pm 1$ and $\mathbf{\Pi}^B_{\uparrow x}$ and $\mathbf{\Pi}^B_{\downarrow x}$ projecting onto states with $\sigma_x^B = \pm 1$. Explicitly, they are

$$\begin{aligned}
\mathbf{\Pi}^A_{\uparrow y} &= \left(|\uparrow\rangle_A + i|\downarrow\rangle_A\right)\left({}_A\langle\uparrow| - i\,{}_A\langle\downarrow|\right)/2\,, \\
\mathbf{\Pi}^A_{\downarrow y} &= \left(|\uparrow\rangle_A - i|\downarrow\rangle_A\right)\left({}_A\langle\uparrow| + i\,{}_A\langle\downarrow|\right)/2\,, \\
\mathbf{\Pi}^B_{\uparrow x} &= \left(|\uparrow\rangle_B + |\downarrow\rangle_B\right)\left({}_B\langle\uparrow| + {}_B\langle\downarrow|\right)/2\,, \\
\mathbf{\Pi}^B_{\downarrow x} &= \left(|\uparrow\rangle_B - |\downarrow\rangle_B\right)\left({}_B\langle\uparrow| - {}_B\langle\downarrow|\right)/2\,.
\end{aligned}$$

The weak values of the products $\mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\uparrow x}$, $\mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\downarrow x}$, $\mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\uparrow x}$ and $\mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\downarrow x}$ are

$$\begin{aligned}
\langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w &= -1/2\,, \\
\langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w &= 1/2\,, \\
\langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w &= 1/2\,, \\
\langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w &= 1/2\,.
\end{aligned} \tag{17.5}$$

Hence for Alice's systems, we have

$$\begin{aligned}
\langle \mathbf{\Pi}^A_{\uparrow y}\rangle_w &= \langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w + \langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w = 0\,, \\
\langle \mathbf{\Pi}^A_{\downarrow y}\rangle_w &= \langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w + \langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w = 1\,,
\end{aligned}$$

and so the weak values conform to the constraint $\sigma_y^A = -1$. Similarly, for Bob's systems we have

$$\begin{aligned}
\langle \mathbf{\Pi}^B_{\uparrow x}\rangle_w &= \langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w + \langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w = 0\,, \\
\langle \mathbf{\Pi}^B_{\downarrow x}\rangle_w &= \langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w + \langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w = 1\,,
\end{aligned}$$

and so the weak values conform to the constraint $\sigma_x^B = -1$. Finally, we compute the weak value of $\sigma_y^A\sigma_x^B$ to get

$$\begin{aligned}
\langle \sigma_y^A\sigma_x^B\rangle_w &= \left\langle (\mathbf{\Pi}^A_{\uparrow y} - \mathbf{\Pi}^A_{\downarrow y})(\mathbf{\Pi}^B_{\uparrow x} - \mathbf{\Pi}^B_{\downarrow x})\right\rangle_w \\
&= \langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w - \langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w - \langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w + \langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w \\
&= -1\,;
\end{aligned}$$

so this weak value conforms to the constraint $\sigma_y^A\sigma_x^B = -1$. Note that the projectors $\mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\uparrow x}$ and $\mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\downarrow x}$ correspond to $\sigma_y^A\sigma_x^B = 1$, but the negative weak value $\langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w$ exactly cancels the positive weak value $\langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w$, leaving $\langle \sigma_y^A\sigma_x^B\rangle_w = -\langle \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\downarrow x}\rangle_w - \langle \mathbf{\Pi}^A_{\downarrow y}\mathbf{\Pi}^B_{\uparrow x}\rangle_w = -1$. Thus weak values reconcile all the constraints. Indeed, instead of calculating the weak values in Eq. (17.5) from $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$, we could have obtained them as the solutions of the three constraints $\langle \sigma_y^A\rangle_w = -1$, $\langle \sigma_x^B\rangle_w = -1$ and $\langle \sigma_y^A\sigma_x^B\rangle_w = -1$.

In our calculations, we took $|\Psi_{fin}\rangle$ of Eq. (17.2) to be the postselected state. But we could have chosen any product state as the postselected state. In particular, we could take $|\Psi_{fin}\rangle$ to

be any state postselected from Alice's measurements of σ_x^A or σ_y^A and Bob's measurements of σ_x^B or σ_y^B. Then, by folding the weak values between $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$ with the probability of postselecting $|\Psi_{fin}\rangle$, we would obtain "probabilities" for all four observables that Alice and Bob might measure, whether or not they in fact measure them. The probabilities would not be limited to the interval $[0, 1]$ however. (See Prob. 17.6.)

Negative probabilities are not new; Wigner was the first to apply them to quantum theory [4]. However, negative probabilities have never had implications for measurements. Here they arise in weak values, measurable in weak measurements. Still, we must interpret these negative weak values with care. Negative probabilities do not make quantum mechanics more reasonable. So we do not interpret $\langle \mathbf{\Pi}^A_{\uparrow y} \mathbf{\Pi}^B_{\uparrow x} \rangle_w = -1/2$ as a negative probability. Our (minimal) interpretation is that pairs of spin systems can have opposite physical effects, according to the sign of $\langle \mathbf{\Pi}^A_{\uparrow y} \mathbf{\Pi}^B_{\uparrow x} \rangle_w$. In general, physical systems can have opposite physical effects, in weak measurements, according to the signs of the weak values.

Consider the shell game of Sect. 16.5, for example. In the classical shell game, a single coin lurks under one of three shells; the occupation is 1 for one shell and 0 for the two others. In the quantum shell game (with pre- and postselection), the weak "occupation" is -1 for one shell and 1 for the two others. In both the classical and the quantum games, the total occupation is 1. But in the quantum game, a *positively* charged coin with weak value $\langle \mathbf{\Pi}_C \rangle_w = -1$ repels passing electrons as if it were a *negatively* charged coin in shell C. How many coins are there in the quantum game – one, or three? There is only one coin, since the weak negative charge in shell C repels passing electrons, but does not attract the other two shells (which themselves attract passing electrons). Yet this one coin can attract *and* repel electrons.

17.4 Galilean Dialogue

[Salviati, Sagredo and Simplicio are sitting in a cafe overlooking Tiberias.]

SIMPLICIO. So this guy goes to a psychiatrist and says, "Doc, my brother's crazy – he thinks he's a chicken!" And, uh, the doctor says –

SALVIATI. We all know the joke, Simplicio.[1]

SIMP. Well, when you say that a negative charge can masquerade as a positive charge, or is negative with negative probability, aren't you saying that quantum mechanics is crazy?

SALV. Negative probabilities are unreasonable, I agree. But I can reasonably interpret weak values – even unexpected weak values – according to their effects on measuring devices.

SIMP. You only show how unreasonable quantum mechanics is.

SALV. I show *how* quantum mechanics is unreasonable.

SAGREDO. Perhaps, Simplicio, I can help you understand what Salviati is saying. The guy in the joke is certainly crazy, yet it's not certain that the guy's brother is crazy. Likewise, the fact that quantum mechanics drives us crazy does not prove that quantum mechanics is crazy.

SALV. It drives us crazy because we still think classically, and it is not classical. We have to change our thinking. We cannot change logic. Logic is independent of, and prior to, experiment. We cannot change the laws of probability either – logic and probability are too closely related.

[1] See Chap. 1.

But weak values can help us stop thinking classically about quantum mechanics. There is a paradox about an electron and a positron in overlapping interferometers –

SAGR. We talked about it.[2]

SALV. Well, the paradox has the following conventional explanation. "We can talk only about what we measure. For example, we can demonstrate that the positron took its overlapping path by placing a sensor on its overlapping path (outside the electron interferometer). The electron can arrive at detector D_- because the positron, by crossing into the electron interferometer, destroys the interference between the overlapping and nonoverlapping electron paths. The positron can arrive at detector D_+ because our sensor destroys the interference between the overlapping and nonoverlapping positron paths. There is no paradox!"

SAGR. And if we don't place any sensor?

SIMP. "We can talk only about what we measure."

SALV. If we think classically, we can talk only about what we measure. But consider the weak values of $\mathbf{\Pi}_{no}^-$, $\mathbf{\Pi}_o^-$, $\mathbf{\Pi}_{no}^+$ and $\mathbf{\Pi}_o^-$ and their products, where $\mathbf{\Pi}_{no}^-$ projects onto the state $|no\rangle_-$, etc. The simultaneous clicking of D_- and D_+ defines the postselected state, and we obtain $\langle \mathbf{\Pi}_{no}^- \mathbf{\Pi}_{no}^+ \rangle_w = -1$, $\langle \mathbf{\Pi}_{no}^- \mathbf{\Pi}_o^+ \rangle_w = 1$, and $\langle \mathbf{\Pi}_o^- \mathbf{\Pi}_{no}^+ \rangle_w = 1$. Also $\langle \mathbf{\Pi}_o^- \mathbf{\Pi}_o^+ \rangle_w$ vanishes, thus the electron and positron never take their overlapping paths *together*. Yet each particle takes its overlapping path *separately*, because $\langle \mathbf{\Pi}_o^- \rangle_w = \langle \mathbf{\Pi}_o^- \mathbf{\Pi}_{no}^+ \rangle_w + \langle \mathbf{\Pi}_o^- \mathbf{\Pi}_o^+ \rangle_w = 1$ and $\langle \mathbf{\Pi}_o^+ \rangle_w = \langle \mathbf{\Pi}_{no}^- \mathbf{\Pi}_o^+ \rangle_w + \langle \mathbf{\Pi}_o^- \mathbf{\Pi}_o^+ \rangle_w = 1$.

SIMP. What are you saying, that *one* electron-positron pair can show up simultaneously, in various guises, on *three* different pairs of paths? I can't accept that.

SALV. Can't you? Well, then, can you accept that *one* positively charged coin can show up simultaneously, in various guises, under *three* shells?[3]

SIMP. Yes, I can. But now I see no reason to accept one and not the other.

17.5 Complex Weak Values

Weak values need not be real. So far, in applying weak values to entanglement, we have encountered real weak values. But Sect. 16.5 gives an example of an imaginary weak projector in the quantum shell game. Here we present an example of how imaginary weak values and entanglement can go together. The example – a calculation of the vector potential of a dipole of imaginary strength – is at first puzzling and even seems to contradict relativistic causality. Yet there is no contradiction.

Suppose two equal and opposite charges separate only at time $t = 0$, making an instantaneous dipole. We can write the charge and current densities of the dipole as follows [5]:

$$\begin{aligned} \rho(\mathbf{x}, t) &= e\sigma_z \delta(x)\delta(y)\delta'(z)\delta(t) \ , \\ J_x(\mathbf{x}, t) &= 0 \ , \\ J_y(\mathbf{x}, t) &= 0 \ , \\ J_z(\mathbf{x}, t) &= -e\sigma_z \delta(x)\delta(y)\delta(z)\delta'(t) \ . \end{aligned} \tag{17.6}$$

The spin operator σ_z governs the direction of the dipole (which charge moves up the z-axis and which moves down) and e is the magnitude of the charge. Equation (17.6) obeys the continuity

[2] See Sect. 17.1.

[3] See Sect. 16.5.

equation,[4] $\nabla \cdot \mathbf{J} + \partial\rho/\partial t = 0$. We can pre- and postselect the states of the spin to get real and imaginary weak values $\langle\sigma_z\rangle_w$. Then, since this weak value multiplies the charge e, we effectively couple the electromagnetic field to a dipole of real or imaginary strength.

What is the electromagnetic field of the dipole? The Hamiltonian H for the electromagnetic field coupled to localized charge and current densities is (in the Coulomb gauge)

$$\begin{aligned} H = & \sum_{\mathbf{k}} \hbar\omega \mathbf{a}_{\mathbf{k}}^\dagger \cdot \mathbf{a}_{\mathbf{k}} \\ & - \frac{2\sqrt{\pi\hbar}}{L^{3/2}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega}} \int \left[\mathbf{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \mathbf{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \cdot \mathbf{J}(\mathbf{x}) d^3x + \frac{1}{2}\int \rho V d^3x \ , \end{aligned} \tag{17.7}$$

with $\omega = ck$; we neglect the ground state energy of the modes. (See Eq. (8.19) and Prob. 8.10.) In the Coulomb gauge, V is not a degree of freedom; it is the instantaneous Coulomb potential of the charge distribution $\rho(\mathbf{x}, t)$. The degrees of freedom are the modes $\mathbf{a}_{\mathbf{k}}$. From Eqs. (17.6–7) we obtain the Heisenberg equation of motion for $\mathbf{a}_{\mathbf{k}}$:

$$\begin{aligned} \frac{d}{dt}\mathbf{a}_{\mathbf{k}} &= i[H, \mathbf{a}_{\mathbf{k}}]/\hbar \\ &= -i\omega\mathbf{a}_{\mathbf{k}} - i(2\pi/\hbar\omega L^3)^{1/2}\left[\int \mathbf{J}(\mathbf{x}) \cdot \mathbf{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x \ , \ \mathbf{a}_{\mathbf{k}}\right] \\ &= -i\omega\mathbf{a}_{\mathbf{k}} - i\delta'(t) e\sigma_z (2\pi/\hbar\omega L^3)^{1/2}(\hat{\mathbf{z}} - \hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{k}}) \ . \end{aligned} \tag{17.8}$$

The mode $\mathbf{a}_{\mathbf{k}}^\dagger$ is transverse ($\mathbf{a}_{\mathbf{k}}^\dagger \cdot \mathbf{k} = 0$), and the transverse part of $\hat{\mathbf{z}}$ is $\hat{\mathbf{z}} - \hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{k}}$. Hence the appearance of $\hat{\mathbf{z}} - \hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{k}}$ instead of $\hat{\mathbf{z}}$ in Eq. (17.8). If we assume that $\mathbf{a}_{\mathbf{k}}$ vanishes for times $t < 0$, the solution of Eq. (17.8) is

$$\mathbf{a}_{\mathbf{k}}(t) = [-i\delta(t) - \omega e^{-i\omega t}\Theta(t)] e\sigma_z (2\pi/\hbar\omega L^3)^{1/2}(\hat{\mathbf{z}} - \hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{k}}) \ ,$$

where $\Theta(t)$ is the Heaviside function $\Theta(t) = 1/2 + t/2|t|$ and $d\Theta/dt = \delta(t)$. (See Prob. 17.7.)

We can now calculate $\mathbf{A}(\mathbf{x})$ by substituting $\mathbf{a}_{\mathbf{k}}(t)$ into the Fourier expansion for $\mathbf{A}(\mathbf{x})$:

$$\mathbf{A}(\mathbf{x}) = \frac{2c\sqrt{\pi\hbar}}{L^{3/2}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega}} \left[\mathbf{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \mathbf{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \ .$$

(See Prob. 8.10(a).) The modes $\mathbf{a}_{\mathbf{k}}$ and $\mathbf{a}_{\mathbf{k}}^\dagger$ contain the operator σ_z which, we assume, has weak value $\langle\sigma_z\rangle_w$. We will calculate $\mathbf{A}(\mathbf{x})$ for $\langle\sigma_z\rangle_w$ real and for $\langle\sigma_z\rangle_w$ imaginary. For $\langle\sigma_z\rangle_w$ real and $t > 0$ we have

$$\mathbf{A}(\mathbf{x}, t) = -\frac{4\pi c e\langle\sigma_z\rangle_w}{L^3} \sum_{\mathbf{k}} (\hat{\mathbf{z}} - \hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{k}}) \cos(\omega t) e^{i\mathbf{k}\cdot\mathbf{x}} \ . \tag{17.9}$$

[4] Equation (17.6) leads to divergences, artifacts of the δ-functions in $\rho(\mathbf{x}, t)$ and $J_z(\mathbf{x}, t)$. We can avoid these divergences by defining $\delta(x)\delta(y)\delta(z)\delta(t) = (c/\pi^2 D^4) e^{-(x^2+y^2+z^2+c^2t^2)/D^2}$ for nonzero D.

The limit $L \to \infty$ turns the sum $L^{-3}\sum_{\mathbf{k}}$ into the integral $(2\pi)^{-3}\int d^3k$, and

$$\begin{aligned}
\mathbf{A}(\mathbf{x},t) &= -\frac{ce\langle\sigma_z\rangle_w}{2\pi^2}\int(\hat{\mathbf{z}}-\hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{k}})\cos(ckt)e^{i\mathbf{k}\cdot\mathbf{x}}d^3k\ , \\
&= \frac{ce\langle\sigma_z\rangle_w}{2\pi^2}\left(\frac{\hat{\mathbf{z}}}{c^2}\frac{\partial}{\partial t^2}-\nabla\frac{\partial}{\partial z}\right)\int\frac{\cos(ckt)}{k^2}e^{i\mathbf{k}\cdot\mathbf{x}}d^3k\ , \\
&= \frac{ce\langle\sigma_z\rangle_w}{2\pi^2}\left(\frac{\hat{\mathbf{z}}}{c^2}\frac{\partial}{\partial t^2}-\nabla\frac{\partial}{\partial z}\right)2\pi\int_0^\infty\int_0^\pi\cos(ckt)e^{ikr\cos\theta}\sin\theta d\theta dk\ , \\
&= \frac{2ce\langle\sigma_z\rangle_w}{\pi}\left(\frac{\hat{\mathbf{z}}}{c^2}\frac{\partial}{\partial t^2}-\nabla\frac{\partial}{\partial z}\right)\int_0^\infty\cos(ckt)\frac{\sin(kr)}{kr}dk\ ,
\end{aligned}$$

where $r = |\mathbf{x}|$. Integration yields

$$\mathbf{A}(\mathbf{x},t) = ce\langle\sigma_z\rangle_w\left(\frac{\hat{\mathbf{z}}}{c^2}\frac{\partial}{\partial t^2}-\nabla\frac{\partial}{\partial z}\right)\frac{1}{r}\Theta(r-ct)\ . \tag{17.10}$$

Equation (17.10) is, indeed, the classical solution. The solution is causal: the fields $\mathbf{E} = -\nabla V - (1/c)\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\mathbf{A}$ are nonzero only for $r = ct$.

But if $\langle\sigma_z\rangle_w$ is imaginary, then instead of Eq. (17.9) we have

$$\mathbf{A}(\mathbf{x},t) = -i\frac{4\pi ce\langle\sigma_z\rangle_w}{L^3}\sum_{\mathbf{k}}(\hat{\mathbf{z}}-\hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{k}})\sin(\omega t)e^{i\mathbf{k}\cdot\mathbf{x}}\ . \tag{17.11}$$

Equation (17.11) differs from Eq. (17.9) only in the replacement of $\cos(\omega t)$ with $i\sin(\omega t)$. So we immediately have

$$\mathbf{A}(\mathbf{x},t) = i\frac{2ce\langle\sigma_z\rangle_w}{\pi}\left(\frac{\hat{\mathbf{z}}}{c^2}\frac{\partial}{\partial t^2}-\nabla\frac{\partial}{\partial z}\right)\int_0^\infty\sin(ckt)\frac{\sin(kr)}{kr}dk\ ,$$

which integrates to [6]

$$\mathbf{A}(\mathbf{x},t) = ice\langle\sigma_z\rangle_w\left(\frac{\hat{\mathbf{z}}}{c^2}\frac{\partial}{\partial t^2}-\nabla\frac{\partial}{\partial z}\right)\frac{1}{\pi r}\ln\left|\frac{r+ct}{r-ct}\right|\ . \tag{17.12}$$

The minor change from $\cos(\omega t)$ to $i\sin(\omega t)$ implies a major change in $\mathbf{A}(\mathbf{x},t)$: Eq. (17.12) shows that $\mathbf{A}(\mathbf{x},t)$ is *noncausal*, i.e. it is nonzero both inside and outside the light cone of the dipole. The same is true of $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$. For $t < 0$ the field of the imaginary dipole vanishes everywhere, but for $t > 0$, it does not vanish anywhere. The imaginary dipole violates relativistic causality!

This violation is a paradox. We used simple quantum electrodynamics to calculate the dipole field. Quantum electrodynamics is relativistic and causal, so how could the calculation violate relativistic causality? Perhaps the violation disappears when we average over post-selections. For example, let the initial spin state be $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. If the final state is $(|\uparrow\rangle + i|\downarrow\rangle)/\sqrt{2}$, the weak value of σ_z is $\langle\sigma_z\rangle_w = i$. If the final state is $(i|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, then $\langle\sigma_z\rangle_w = -i$ and $\mathbf{A}(\mathbf{x},t)$ is the negative of Eq. (17.12). Since the probability of both final states is 1/2, the field $\mathbf{A}(\mathbf{x},t)$ vanishes when we average over them. Yet averaging does not resolve the paradox of a *single* imaginary dipole.

Relativistic causality forbids superluminal messages. Suppose that Alice, in her laboratory, creates an imaginary dipole via pre- and postselection. Bob, in his laboratory, can measure the field $\mathbf{A}(\mathbf{x},t)$ at a spacelike separation from the dipole. Can Alice send a superluminal message to Bob? She cannot, and the reason is similar to the reason that the Cherenkov radiation of a charge moving with weak superluminal speed is consistent with causality. (See Sect. 16.7 and Prob. 16.16.) Alice obtains the weak value $\langle\sigma_z\rangle_w = i$ only if any measurement that couples to σ_z is also weak. If e, the magnitude of the charge, is too large, the Hamiltonian H in Eq. (17.7) will not contain $\langle\sigma_z\rangle_w$ in the current $J_z(\mathbf{x},t)$, but rather one of the eigenvalues of σ_z. Then any noncausal field that Bob measures must be a fluctuation of the vacuum. Indeed, the field and the vacuum are not orthogonal, when both are restricted to the finite spacetime region that Bob measures [7]. If e is not too large, then H contains $\langle\sigma_z\rangle_w = i$ and Bob measures the noncausal field $\mathbf{A}(\mathbf{x},t)$ in Eq. (17.12). But Bob cannot tell from his measurements what Alice did in her laboratory; as far as he knows, the field he measures could be a fluctuation of the vacuum. The nonlocal correlations in $\mathbf{A}(\mathbf{x},t)$ show up only when he and Alice compare their measurements.

Is this resolution complete? The field $\mathbf{A}(\mathbf{x},t)$ of an imaginary dipole is noncausal, while the field of a real dipole is causal. If noncausal fields are useless for sending superluminal messages, why are the fields of real dipoles causal? The complete resolution of the paradox takes unitarity into account [8]. In Eq. (17.7), H becomes an effective Hamiltonian H_{eff} when we substitute $\langle\sigma_z\rangle_w$ for σ_z. If $\langle\sigma_z\rangle_w$ is real, H_{eff} generates unitary evolution. If $\langle\sigma_z\rangle_w$ is not real, H_{eff} generates nonunitary evolution, which can alter nonlocal correlations. (See Sect. 16.5 and Probs. 16.9–10.) A simple example involves two photons on a line, with coordinates x_1 and x_2, in an entangled state

$$\left[\psi(x_1)\psi(x_2-L)+\psi(x_1-a)\psi(x_2-L-a)\right]/\sqrt{2}\,,$$

where $0 < a \ll L$; assume $\psi(x)$ vanishes unless $x \approx 0$. Measurements on this state show $x_1 \approx 0$, $x_2 \approx L$ or, with equal probability, $x_1 \approx a$, $x_2 \approx L+a$. Suppose the effective Hamiltonian for the two photons is $H_{eff} = \langle\sigma_z\rangle_w x_1$, where x_1 is the position operator for the first photon. Then the state of the photons at time t is

$$e^{i\langle\sigma_z\rangle_w x_1 t}\left[\psi(x_1)\psi(x_2-L)+\psi(x_1-a)\psi(x_2-L-a)\right]/\sqrt{2}\,, \tag{17.13}$$

If $\langle\sigma_z\rangle_w$ is real, the relative phase of the terms in Eq. (17.13) changes over time, but measurements still show $x_1 \approx 0$, $x_2 \approx L$ or $x_1 \approx a$, $x_2 \approx L+a$ with equal probability. But if $\langle\sigma_z\rangle_w$ is not real, Eq. (17.13) approaches either $\psi(x_1)\psi(x_2-L)$ or $\psi(x_1-a)\psi(x_2-L-a)$. Thus a complex effective Hamiltonian can change probabilities nonlocally, just as a measurement on one photon can change probabilities for the other. Neither the complex Hamiltonian nor the photon measurement violates causality; but applied to entangled states, both can change probabilities nonlocally. (See also Prob. 3.11.)

This explanation might not seem to apply to the imaginary dipole. The vacuum state $|0\rangle$ is a product state: it is the product of the ground states of all the modes $a_{\mathbf{k},i}$. But these ground states are nonlocal (momentum) states, extending over all of space. In a basis of *local* (position) states, the vacuum is a highly entangled state. To check that the vacuum is entangled in position, we can calculate the spatial correlations of $\mathbf{A}$. We define

$$\mathcal{A}(\mathbf{x}_1) = \pi^{-3/2}D^{-3}\int e^{-(\mathbf{x}-\mathbf{x}_1)^2/D^2}\mathbf{A}(\mathbf{x})d^3x\,, \tag{17.14}$$

and calculate the vacuum correlation of $\mathcal{A}(\mathbf{x}_1)$ and $\mathcal{A}(\mathbf{x}_2)$. (The vacuum correlation of $\mathbf{A}(\mathbf{x}_1)$ and $\mathbf{A}(\mathbf{x}_2)$ diverges, hence we consider the correlation of $\mathcal{A}(\mathbf{x}_1)$ and $\mathcal{A}(\mathbf{x}_2)$ instead.) Since $\langle 0|\mathcal{A}(\mathbf{x}_1)|0\rangle = 0 = \langle 0|\mathcal{A}(\mathbf{x}_2)|0\rangle$, the vacuum correlation is

$$\langle 0|\mathcal{A}(\mathbf{x}_1)\mathcal{A}(\mathbf{x}_2)|0\rangle \ ,$$

and this vacuum correlation does not vanish as it would if the vacuum were a product of local position states. (See Prob. 17.10.)

Problems

17.1 Section 6.4 defines a parity operator P for a photon passing through the Mach-Zehnder interferometer. For the electron and positron in Sect. 17.1, define two analogous parity operators, P_- and P_+, respectively. Let the initial state of the particles be

$$\frac{1}{2}\left[e^{i\alpha_+}|no\rangle_+ + |o\rangle_+\right] \otimes \left[e^{i\alpha_-}|o\rangle_- + |no\rangle_-\right]$$

in the notation of Sect. 17.1.
(a) Show that the expectation values of P_-, P_+ and P_-P_+ in the initial state are

$$\langle P_-\rangle = \cos\alpha_- \ , \quad \langle P_+\rangle = \cos\alpha_+ \ , \quad \langle P_-P_+\rangle = \cos\alpha_- \cos\alpha_+ \ .$$

(b) Show that if the "bomb" does not explode, the corresponding expectation values for the particles are

$$\langle P_-\rangle = \frac{2}{3}\cos\alpha_- \ , \quad \langle P_+\rangle = \frac{2}{3}\cos\alpha_+ \ , \quad \langle P_-P_+\rangle = \frac{2}{3}\cos(\alpha_- + \alpha_+) \ .$$

*17.2 (a) For the states $|\Psi_{in}\rangle$, $|\Psi_{fin}\rangle$ and projectors $\mathbf{\Pi_1}$, $\mathbf{\Pi_2}$ of Sect. 17.2, apply the ABL formula, Eq. (10.8), to show that an intermediate measurement of the location of the Cat (or neutron) will find it in the state $|2\rangle$, and an intermediate measurement of its grin (or spin) will find it in the state $|\uparrow\rangle$.
(b) Consider a particle with two internal spin-1/2 degrees of freedom, and corresponding operators σ_z and σ'_z. Let $\mathbf{\Pi_1}$ and $\mathbf{\Pi_2}$ project the particle onto localized states $|1\rangle$ and $|2\rangle$, respectively. Find states $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$ satisfying the following constraints: after preselection of $|\Psi_{in}\rangle$ and before postselection of $|\Psi_{fin}\rangle$, the particle is certainly localized to $|1\rangle$ and certainly has $\sigma_z = \sigma'_z = 1$ (according to the ABL formula); but the weak values of $\sigma_z\mathbf{\Pi_1}$ and $\sigma'_z\mathbf{\Pi_1}$ vanish. (Assume that σ_z, σ'_z, $\mathbf{\Pi_1}$ and $\mathbf{\Pi_2}$ all commute with the Hamiltonian of the particle.)

17.3 Calculate $\langle\mathbf{\Pi_1\Pi_\uparrow}\rangle_w$, $\langle\mathbf{\Pi_1\Pi_\downarrow}\rangle_w$, $\langle\mathbf{\Pi_2\Pi_\uparrow}\rangle_w$ and $\langle\mathbf{\Pi_2\Pi_\downarrow}\rangle_w$ for the states $|\Psi_{in}\rangle$, $|\Psi_{fin}\rangle$ and projectors $\mathbf{\Pi_1}$, $\mathbf{\Pi_2}$, $\mathbf{\Pi_\downarrow}$ and $\mathbf{\Pi_\uparrow}$ of Sect. 17.2, using only the four weak values $\langle\sigma_z\rangle_w = 1$, $\langle\mathbf{\Pi_1}\rangle_w = 0$, $\langle\mathbf{\Pi_2}\rangle_w = 1$ and $\langle\mathbf{\Pi_2}\sigma_z\rangle_w = 0$.

17.4 (a) Calculate the weak values in Eq. (17.5) using the states $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$ of Sect. 17.3.
(b) Now obtain the weak values in Eq. (17.5) using only the three weak values $\langle\sigma_y^A\rangle_w = -1$, $\langle\sigma_x^B\rangle_w = -1$ and $\langle\sigma_y^A\sigma_x^B\rangle_w = -1$.

17.5 Suppose the initial state $|\Psi_{in}\rangle$ of three spins is the GHZ state

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_A \otimes |\uparrow\rangle_B \otimes |\uparrow\rangle_C - |\downarrow\rangle_A \otimes |\downarrow\rangle_B \otimes |\downarrow\rangle_C]$$

and the final state $|\Psi_{fin}\rangle$ has all three spins pointing down the x-axis:

$$|\Psi_{fin}\rangle = \frac{1}{2\sqrt{2}}[|\uparrow\rangle_A - |\downarrow\rangle_A] \otimes [|\uparrow\rangle_B - |\downarrow\rangle_B] \otimes [|\uparrow\rangle_C - |\downarrow\rangle_C] \, .$$

The GHZ state $|\Psi_{in}\rangle$ is an eigenstate of the observables $\sigma_x^A \sigma_y^B \sigma_y^C$, $\sigma_y^A \sigma_x^B \sigma_y^C$ and $\sigma_y^A \sigma_y^B \sigma_x^C$ with eigenvalue 1.
(a) Prove

$$\langle \sigma_y^A \sigma_y^B \rangle_w = \langle \sigma_y^B \sigma_y^C \rangle_w = \langle \sigma_y^A \sigma_y^C \rangle_w = -1 \, . \tag{17.15}$$

(b) Define projectors $\mathbf{\Pi}^A_{\uparrow y}$, $\mathbf{\Pi}^A_{\downarrow y}$ according to

$$\mathbf{\Pi}^A_{\uparrow y} = (|\uparrow\rangle_A + i|\downarrow\rangle_A)\,({}_A\langle\uparrow| - i_A\langle\downarrow|)/2 \, ,$$
$$\mathbf{\Pi}^A_{\downarrow y} = (|\uparrow\rangle_A - i|\downarrow\rangle_A)\,({}_A\langle\uparrow| + i_A\langle\downarrow|)/2 \, ,$$

and likewise for the other two spins. Define also $\mathbf{\Pi}_{+++} = \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\uparrow y}\mathbf{\Pi}^C_{\uparrow y}$, likewise $\mathbf{\Pi}_{++-} = \mathbf{\Pi}^A_{\uparrow y}\mathbf{\Pi}^B_{\uparrow y}\mathbf{\Pi}^C_{\downarrow y}$ and so on. Equation (17.15) is a constraint on the eight weak values $\langle\mathbf{\Pi}_{+++}\rangle_w$, $\langle\mathbf{\Pi}_{++-}\rangle_w$, …, $\langle\mathbf{\Pi}_{---}\rangle_w$. Evaluate the eight weak values using only Eq. (17.15) and symmetry considerations.

17.6 Alice and Bob share pairs of subsystems in an entangled state $|\Psi\rangle$; Alice measures either A or A' on one subsubsystem of each pair while, far away, Bob measures either B or B' on the other. Assume A, A', B and B' have finitely many eigenvalues a_i, a'_j, b_k and b'_l respectively, all nondegenerate and bounded by 1 in absolute magnitude, and $[A, A'] \neq 0 \neq [B, B']$. Section 3.4 defines local plans λ with a probability measure $\rho(\lambda)d\lambda$ such that $\int \rho(\lambda)d\lambda = 1$. Let us define local plans λ_{ijkl} such that a measurement of A, A', B or B' on a pair carrying local plan λ_{ijkl} yields, respectively, a_i, a'_j, b_k or b'_l, i.e.

$$P(A; a_m; \lambda_{ijkl}) = \delta_{im} \, , \quad P(B; b_n; \lambda_{ijkl}) = \delta_{kn} \, ,$$

and

$$P(A, B; a_m, b_n; \lambda_{ijkl}) = P(A; a_m; \lambda_{ijkl})P(B; b_n; \lambda_{ijkl}) = \delta_{im}\delta_{kn} \, .$$

(Compare Eq. (3.10).) Define $\rho(\lambda_{ijkl})$ to be $|\langle a_i, b_k|\Psi\rangle|^2$ times the weak value of $|a'_j, b'_l\rangle\langle a'_j, b'_l|$ between the preselected state $|\Psi\rangle$ and the postselected state $|a_i, b_k\rangle$. Show that

$$\sum_{ijkl} \rho(\lambda_{ijkl}) = 1$$

and that $P(A, B; a_m, b_n)$,

$$P(A, B; a_m, b_n) = \sum_{ijkl} P(A, B; a_m, b_n; \lambda_{ijkl})\rho(\lambda_{ijkl}) \ ,$$

reproduces the quantum probability that measurements of A and B yield a_m and b_n, respectively. The local plans λ_{ijkl} reproduce all the quantum probabilities for measurements of A, A', B and B' on $|\Psi\rangle$, in apparent violation of Bell's theorem. (See Sect. 3.4.) What is the catch?

17.7 A one-dimensional harmonic oscillator with a driving function $\lambda f(t)$ has a Hamiltonian

$$H = (a^\dagger a + 1/2)\hbar\omega + \lambda f(t)(a^\dagger + a)\sqrt{\hbar/2\omega} \ .$$

(a) Show that the solution of the Heisenberg equation of motion for a is

$$a(t) = a_0(t) - i\lambda\sqrt{\hbar/2\omega}\int_{-\infty}^{t} f(t')e^{-i\omega(t-t')}dt' \ ,$$

where $a_0(t)$ is any solution of the equation of motion for $\lambda = 0$.
(b) If $f(t) = \delta'(t)$ and $a(t) = 0$ for $t < 0$, show that

$$a(t) = -i\delta(t)\lambda\sqrt{\hbar/2\omega} - \lambda\sqrt{\omega/2\hbar}e^{-i\omega t}\Theta(t) \ .$$

*17.8 Compute $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ from $\mathbf{A}(\mathbf{x}, t)$ in Eq. (17.10) and the potential V due to the instantaneous charge density $\rho(\mathbf{x}, t)$ in Eq. (17.6).

17.9 Starting from the Hamiltonian H in Eq. (8.19), the charge and current densities in Eq. (17.6), and the commutation relation $[A_i^{(\mathbf{k})}, \Pi_j^{(\mathbf{k}')}] = i\hbar(\delta_{ij} - k_i k_j/k^2)\delta^{\mathbf{k},\mathbf{k}'}$, derive the Heisenberg equation of motion for $\mathbf{A}^{(\mathbf{k})}$:

$$\frac{d^2}{dt^2}\mathbf{A}^{(\mathbf{k})} = -\omega^2\mathbf{A}^{(\mathbf{k})} - \frac{2\sqrt{\pi}e\sigma_z}{L^{3/2}}\delta'(t)(\hat{\mathbf{z}} - \hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\hat{\mathbf{k}}) \ .$$

Solve the equation of motion assuming that $\mathbf{A}^{(\mathbf{k})}$ vanishes for $t < 0$.

17.10 Show that the vacuum correlation

$$\langle 0|\mathcal{A}(\mathbf{x}_1)\mathcal{A}(\mathbf{x}_2)|0\rangle$$

equals $(2\hbar c/\pi r)\int_0^\infty e^{-k^2D^2/2}\sin(kr)dk$, where $r = |\mathbf{x}_1 - \mathbf{x}_2|$. (See Eq. (17.14) for the definition of $\mathcal{A}(\mathbf{x}_1)$.)

*17.11 Consider a weak measurement, on the PPS ensemble of Sect. 17.1, of the Coulomb force between an electron and a positron on the non-overlapping paths of Fig. 17.1. Show that the weak force is repulsive, although the electron and positron have opposite charges. (This thought experiment may be feasible with trapped ions [9] or photons [10] replacing the electron and positron.)

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18 The Quantum World

Planck's 1900 discovery of quanta opened a "quantum century" of fundamental discoveries in quantum mechanics. Now a second "quantum century" has opened:

> In the nineteenth century, life was transformed by the conscious application of classical mechanics, in the form of Newton's equations (and, later, thermodynamics), to the engines of the industrial revolution. In this century, a similar transformation has been wrought by electromagnetism, in generating and distributing electric power and communicating words and pictures across the world at the speed of light, in what should be seen as a conscious application of Maxwell's equations. It is easy to predict that in the twenty-first century it will be quantum mechanics that influences all our lives [1].

This final chapter looks back on the first quantum century and looks forward, not to applications, but to fundamental discoveries in the second quantum century.

One fundamental discovery was that many measurements necessarily disturb the measured system. But measurements of an "eigenoperator" of the system do not; and other measurements that do not disturb the measured system include protective measurements and weak measurements. (See Chaps. 15–17). One theme of this book is that it is particularly *these* measurements – measurements that do not disturb the measured system – that develop our intuition of quantum mechanics. The weak measurements in the next two sections develop this intuition further.

A second theme of the book is its demand for simple and physical axioms for quantum mechanics. Chapter 6 suggests two axioms: causality and nonlocality. Quantum mechanics may not follow from these two axioms alone. But, starting with Sect. 18.2, this chapter suggests simple, physical axioms for quantum theory.

18.1 Weak Measurements and Interference

Sections 2.4, 4.1, 15.1 and 16.1 present thought experiments – variations on the two-slit interference experiment – that challenge complementarity. They get more and more sophisticated, but they all fail: whenever they show which way the particles go, they fail to show an interference pattern, and vice versa. This section presents our most sophisticated thought experiment yet – and it does not fail! It shows through which slit each particle goes, and it shows an interference pattern.

The thought experiment includes a weak measurement of an expectation value on a PPS ensemble. What is the connection between a weak value and an expectation value? Suppose

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we have an ensemble of N identical systems in the same initial state $|\psi\rangle$. On this ensemble we measure $\bar{A}$, the average value of an operator A. If $A^{(i)}$ represents the observable A on the i-th system, then $\bar{A}$ is

$$\bar{A} = \frac{1}{N}\sum_{i=1}^{N} A^{(i)} \ .$$

The measurement of $\bar{A}$ is naturally weak when N is large, for it is a measurement of A/N on each system. For a weak measurement of $\bar{A}$ on a PPS ensemble, we postselect the states of the N systems. We could postselect improbable states. But this time, after the measurement of $\bar{A}$, we merely measure the state of each system in an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle, \ldots$ and "postselect" whatever basis state we get. Then, since the pre- and postselected states are product states, the weak value of $\bar{A}$ is the average of the weak values of the $A^{(i)}$,

$$\langle \bar{A} \rangle_w = \frac{1}{N}\sum_{i=1}^{N} \langle A^{(i)} \rangle_w \ ,$$

and the weak value $\langle A^{(i)} \rangle_w$ depends only on the initial and final states of the i-th system. For some coefficients c_k the preselected state of the systems is

$$|\psi\rangle = \sum_k c_k |k\rangle \ ,$$

and we postselect N_1 of the systems in the state $|1\rangle$, N_2 of the systems in the state $|2\rangle$, and so on, for some $N_1 + N_2 + \cdots = N$. The weak value of $\bar{A}$ is then

$$\langle \bar{A} \rangle_w = \frac{1}{N}\sum_k N_k \frac{\langle k|A|\psi\rangle}{\langle k|\psi\rangle} = \sum_k \frac{N_k}{N}\frac{\langle k|A|\psi\rangle}{c_k} \ . \tag{18.1}$$

Now, the most probable postselected state will have $N_1 = |c_1|^2 N$ systems in the state $|1\rangle$, $N_2 = |c_2|^2 N$ systems in the state $|2\rangle$, and so on.[1] So from Eq. (18.1) we obtain

$$\langle \bar{A} \rangle_w = \sum_k |c_k|^2 \frac{\langle k|A|\psi\rangle}{c_k} = \sum_k c_k^* \langle k|A|\psi\rangle = \langle \psi|A|\psi\rangle \tag{18.2}$$

as the most probable outcome of the measurement of $\bar{A}$; and the larger N, the smaller are the expected corrections to Eq. (18.2). That is, the *weak* value of $\bar{A}$ approaches the *expectation* value of A in the initial state $|\psi\rangle$.

This connection between a weak value and an expectation value seems straightforward. But now consider electrons in a two-slit experiment. The state of each electron at time $t = 0$, just as it passes through the screen with the two slits, is $c_L|L\rangle + c_R|R\rangle$, where $|L\rangle$ and $|R\rangle$ are quantum waves emerging from the left and right slits, respectively. At time $t = T$, the electron

[1] The weak measurement of $\bar{A}$ does not change this probability distribution. For large N, the probability that a measurement of $A^{(i)}/N$ changes the state of the i-th system is proportional to $1/N^2$; the probability that it changes the state of any system is therefore proportional to $1/N$, which vanishes in the limit $N \to \infty$.

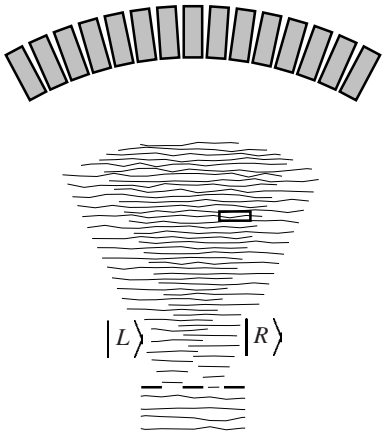

Figure 18.1: Electron waves overlap in a two-slit interference experiment. The row of detectors at the top reveals through which slit the electron passed; the operator A projects the wave onto the small rectangle in the overlap of the waves.

strikes a detector that measures its transverse momentum, thus showing through which slit it passed. (See Fig. 18.1.) That is, the detector projects the state of each electron onto $U(T)|L\rangle$ or $U(T)|R\rangle$, where $U(T)$ represents time evolution from $t = 0$ to $t = T$. At an intermediate time $0 < t < T$ we measure weakly an operator $\bar{A}$ that shows the electron interference pattern. For example, let $A^{(i)}$ project the transverse position of the i-th electron onto a narrow interval. Then $\bar{A} = \sum_i A^{(i)}/N$ is the fraction of the N electrons passing through that transverse interval. Applying Eq. (18.2) to this experiment, we find that the weak value of $\bar{A}$ is the expectation value of A in the state $c_L U(t)|L\rangle + c_R U(t)|R\rangle$; and if the interval is narrow enough, this expectation value will correspond to a dark or light strip in the two-slit interference pattern. We could map out the whole interference pattern by transversely moving the interval onto which A projects, or (for fixed A) by changing the relative phase between $U(t)|L\rangle$ and $U(t)|R\rangle$. Either way, we obtain the interference pattern that corresponds to the state $c_L U(t)|L\rangle + c_R U(t)|R\rangle$, and we also observe through which slit each electron passed.

We can simplify the experiment by restricting it to a line (i.e. to the transverse coordinate). We prepare the first electron on a line in the initial state $c_L|L\rangle + c_R|R\rangle$, where $|L\rangle$ is a wave packet coming from the left, and $|R\rangle$ is a wave packet coming from the right with the opposite momentum. When the two wave packets meet and overlap, we measure their interference pattern via a weak measurement of $A^{(1)}/N$, where $A^{(1)}$ projects the state of the first electron to an interval within the overlap region. After the wave packets separate, we check whether the electron is in the state $|L\rangle$ or $|R\rangle$ (which way it is moving). We then send in the second electron in the same initial state, measure $A^{(2)}/N$, and check whether it is in the state $|L\rangle$ or $|R\rangle$; and so on with the other electrons, until we have measured $\langle\bar{A}\rangle$ on N electrons. We thus measure the final direction of each electron and also the intensity of the superposition $c_L|L\rangle + c_R|R\rangle$ in the interval onto which $\bar{A}$ projects. By moving this interval and measuring again, we eventually map out the entire interference pattern.

Is this thought experiment a paradox? On the one hand, the previous thought experiments confirmed the intuition that "it is impossible to design an apparatus to determine which hole the electron passes through, that will not at the same time disturb the electrons enough to destroy the interference pattern" [2]. On the other hand, we have developed the intuition that a weak measurement on a PPS ensemble tells us about *both* the pre- and the postselected states

– here, about the interference pattern and about the final direction of motion of each electron. The thought experiment (which does not contradict any uncertainty relations, either for the electrons or for the screen) bears out this intuition.

But it is a paradox. Bohr argued that Einstein's thought experiments in Sect. 2.4 must fail – and we argued that the thought experiments of Sects. 4.1, 15.1 and 16.1 must fail – if quantum theory is consistent. How can quantum mechanics be consistent if this experiment succeeds where the others fail? If each electron passes through one slit, how can there be an interference pattern? Suppose we insert a line of magnetic flux between the slits. The magnetic flux shifts the interference pattern. But how can it, if each electron passes only on one side of the flux line? We might suspect that the weak measurement changes the direction of some of the electrons but, as noted above, the probability that the state of any electron changes vanishes in the limit $N \to \infty$.

18.2 From Amplitudes to Probabilities

One of the axioms of quantum theory is the Born [3] probability rule: Let A be a Hermitian operator with eigenvalues a_k and corresponding orthonormal eigenstates $|k\rangle$. The probability that a measurement of A on a system in the state $|\psi\rangle$ yields the eigenvalue a_k is $|\langle k|\psi\rangle|^2$. (For simplicity we take A nondegenerate.) A number of physicists [4] claim to have derived this axiom from a weaker axiom, the axiom of collapse: the measurement of A collapses the state to one of the eigenstates of A. The axiom of collapse is weaker in that it assumes that a measurement of A on N identical systems prepared in the state $|\psi\rangle$ leaves N_1 of them in the state $|1\rangle$, N_2 of them in the state $|2\rangle$, and so on; but it does not assume anything else about the N_k except that $N_1 + N_2 + \cdots = N$. Other physicists [5] dispute this claim. To test this claim we analyze [6] here a weak measurement of A.

As in the previous section, we consider an ensemble of N identical systems preselected in an initial state $|\psi\rangle$, where

$$|\psi\rangle = \sum_k c_k |k\rangle$$

and the $|k\rangle$ are an orthonormal basis of states. The only change is that here we assume that the basis states $|k\rangle$ are eigenstates of A:

$$A|k\rangle = a_k|k\rangle \ .$$

The eigenvalues a_k are still arbitrary. On these N systems we measure $\bar{A}$, which is naturally weak when N is large:

$$\bar{A} = \frac{1}{N}\sum_{i=1}^{N} A^{(i)} \ ;$$

again $A^{(i)}$ represents the observable A on the i-th system. For the weak measurement of $\bar{A}$ we prepare a measuring device in an initial state $|\Phi_d\rangle$ and couple it to the measured systems via an interaction Hamiltonian H_{int},

$$H_{int} = g(t) P_d \bar{A} \ ,$$

where $[Q_d, P_d] = i\hbar$ and Q_d is the displacement of a pointer on the measuring device. It is convenient to let $g(t)$ be impulsive and $\int g(t)dt = 1$. After the weak measurement of $\bar{A}$ we conclude the experiment with a strong (precise) measurement of A on each system. Again we postselect whatever we get, so that the postselected state of the systems is a product state with N_1 systems in the state $|1\rangle$, N_2 systems in the state $|2\rangle$, and so on, where $N_1 + N_2 + \cdots = N$.

Immediately after the weak measurement, the state of the measuring device and systems is

$$e^{-iP_d\bar{A}/\hbar}|\Psi_{in}\rangle \otimes |\Phi_d\rangle \ , \tag{18.3}$$

where we define

$$|\Psi_{in}\rangle = \bigotimes_{i=1}^{N} |\psi\rangle_i \ .$$

In Eq. (18.3) we have neglected the Hamiltonians of the measured systems and the measuring device separately, since $g(t)$ is impulsive. By factoring the time evolution operator, we can rewrite Eq. (18.3) as

$$\left(\bigotimes_{i=1}^{N} e^{-iP_dA^{(i)}/N\hbar}|\psi\rangle_i\right) \otimes |\Phi_d\rangle \ . \tag{18.4}$$

We expand the exponential in Eq. (18.4) in powers of $1/N$:

$$e^{-iP_dA^{(i)}/N\hbar} = 1 - \frac{i}{N\hbar}P_dA^{(i)} + \mathcal{O}\left[\frac{1}{N^2}\right] \ . \tag{18.5}$$

Part (a) of Prob. 3.10 states that

$$A|\psi\rangle = \langle A\rangle|\psi\rangle + \Delta A|\psi_\perp\rangle \ ,$$

where $\langle A\rangle$ is the expectation value of A in the state $|\psi\rangle$, $\Delta A = (\langle A^2\rangle - \langle A\rangle^2)^{1/2}$ and $|\psi_\perp\rangle$ is a normalized state orthogonal to $|\psi\rangle$. Then Eq. (18.5) applied to $|\psi\rangle_i$ yields

$$e^{-iP_dA^{(i)}/N\hbar}|\psi\rangle_i = \left[1 - \frac{i}{N\hbar}P_d\langle A\rangle\right]|\psi\rangle_i - \frac{i}{N\hbar}P_d(\Delta A)|\psi_\perp\rangle_i + \mathcal{O}\left[\frac{1}{N^2}\right] \ . \tag{18.6}$$

For the product state $|\Psi_{in}\rangle$ we obtain, to the same order,

$$e^{-iP_d\bar{A}/\hbar}|\Psi_{in}\rangle = \left[1 - \frac{iP_d\langle A\rangle}{N\hbar}\right]^N \left[|\Psi_{in}\rangle - \frac{iP_d\Delta A}{\hbar\sqrt{N}}\sum_{i=1}^{N}\frac{|\Psi_i^\perp\rangle}{\sqrt{N}}\right] + \mathcal{O}\left[\frac{1}{N^2}\right] \ , \tag{18.7}$$

where $|\Psi_i^\perp\rangle$ represents the i-th system in the state $|\psi_\perp\rangle$ and all other systems in the state $|\psi\rangle$. Since $\langle\Psi_i^\perp|\Psi_j^\perp\rangle = \delta_{ij}$, the expression $\sum_i |\Psi_i^\perp\rangle/\sqrt{N}$ in Eq. (18.7) is normalized, as is $|\Psi_{in}\rangle$. Thus for large N, the sum in Eq. (18.7) is a small perturbation on $|\Psi_{in}\rangle$, and Eq. (18.7) approaches $e^{-iP_d\langle A\rangle/\hbar}|\Psi_{in}\rangle$. The larger N, the weaker the measurement of $\bar{A}$, and the less the measurement disturbs $|\Psi_{in}\rangle$.

After the weak measurement, the state of the systems collapses to a direct product of eigenstates of A; the final state $|\Psi_{fin}\rangle$ is a product state of N_1 systems in the state $|1\rangle$, N_2 systems in the state $|2\rangle$, and so on, where $N_1 + N_2 + \cdots = N$. What is the state of the measuring device? On the one hand, Eq. (18.7) implies that the position of the pointer has changed by $\langle A\rangle$ (if we can neglect the corrections to $e^{-iP_d\bar{A}/\hbar}|\Psi_{in}\rangle$). On the other hand, we can calculate the shift in the pointer directly from the inner product of $|\Psi_{fin}\rangle$ and Eq. (18.3) by applying $e^{iP_d\bar{A}/\hbar}$ to $|\Psi_{fin}\rangle$. Since A takes the value a_1 on N_1 of the systems, the value a_2 in N_2 of the systems, and so on, the pointer shifts by $(N_1a_1 + N_2a_2 + \dots)/N$. These two calculations of the shift must agree, hence

$$\frac{N_1}{N}a_1 + \frac{N_2}{N}a_2 + \cdots = |c_1|^2a_1 + |c_2|^2a_2 + \dots \ ;$$

and since the eigenvalues $a_1, a_2, \dots$ of A are arbitrary, we conclude that $N_1/N = |c_1|^2$, $N_2/N = |c_2|^2$, and so on. The fraction N_k/N of systems on which a measurement of A yields a_k approaches $|c_k|^2$ in the limit $N \to \infty$, and we recover the Born probability rule.

But *can* we neglect of the corrections to $e^{-iP_d\bar{A}/\hbar}|\Psi\rangle$? If, for example, $N_1 = N$ in the final state, we do not recover the Born probability rule. The inner product of this final state with the corrections does not vanish. On the contrary: for final states $|\Psi_{fin}\rangle$ that violate the Born rule, the inner product

$$\sum_{i=1}^{N}\langle\Psi_{fin}|\Psi_i^{\perp}\rangle/\sqrt{N} \tag{18.8}$$

does *not* vanish for large N; instead of cancelling one another, the terms in the sum add coherently and Eq. (18.8) is proportional to $\sqrt{N}$. (See Prob. 18.1.)

Thus the axiom of collapse is not sufficient. What we require is an axiom of *stable* collapse. Our thought experiment proceeds from an initial state $|\Psi_{in}\rangle \otimes |\Phi_d\rangle$ via a weak measurement of $\bar{A}$ to projection onto $|\Psi_{fin}\rangle$. An arbitrarily small perturbation of the initial state should not induce a measurable change in $\bar{A}$. (If it did, we could never measure $\bar{A}$, for an arbitrarily small perturbation of the initial state would change the result of the measurement.) Thus we must assume that the variation in the measured value of $\bar{A}$ due to the variation

$$|\Psi_{in}\rangle \to (1+\epsilon^2)^{-1/2}\left[|\Psi_{in}\rangle + \epsilon|\Psi_{in}^{\perp}\rangle\right]$$

vanishes as ϵ vanishes, for any normalized $|\Psi_{in}^{\perp}\rangle$ orthogonal to $|\Psi_{in}\rangle$.

Now consider the initial state $|\Psi_{in}\rangle = |\psi\rangle_1 \otimes \cdots \otimes |\psi\rangle_N$ expanded in an orthonormal basis of eigenstates of $A^{(1)}, A^{(2)}, \dots, A^{(N)}$ i.e. in an orthonormal basis of such states as $|3\rangle_1 \otimes |5\rangle_2 \otimes \cdots \otimes |2\rangle_N$, etc. For every such basis state there is a set of values for $N_1, N_2, \dots$. Let $|\Psi_{in}^T\rangle$ be a truncation of $|\Psi_{in}\rangle$ to basis states which have $N_i = c_i^*c_iN + \mathcal{O}(\sqrt{N})$ for all i. For N large enough, $|\Psi_{in}^T\rangle$ is an arbitrarily small perturbation of $|\Psi_{in}\rangle$, yet $|\Psi_{in}^T\rangle$ projects onto only those final states $|\Psi_{fin}\rangle$ that are consistent with the Born rule. (See Prob. 9.5.) The axiom of stable collapse then implies that also in our thought experiment, with $|\Psi_{in}\rangle$ and not $|\Psi_{in}^T\rangle$ as the initial state, the final state $|\Psi_{fin}\rangle$ must be consistent with the Born rule. Thus we obtain the Born rule from a weaker axiom, the axiom of stable collapse.

18.3 The Fate of the Universe

Chapter 9 presents the paradox of Schrödinger's cat, and some proposed resolutions. These resolutions, while consistent with experiment, are overgrown with redundant physical quantities. Can we apply Occam's razor to the overgrowth? The most overgrown resolution is the Many Worlds interpretation, with its infinity of worlds. Other resolutions add assumptions to quantum theory without adding anything to its predictions. Collapse and hidden variables resolve the paradox by completing quantum theory (if quantum theory is incomplete) but no one has observed collapse,[2] and hidden variables are in principle unobservable. These additional assumptions are reminiscent of the aether assumption; like the aether, collapse and hidden variables seem to have a place in theory, but no place in experiment. (See Sect. 1.4.)

A different way to resolve the paradox is to impose final boundary conditions on the universe. (See Sects. 9.4 and 10.5.) In effect, we impose final boundary conditions on the Many Worlds interpretation, to postselect just one world. In the Many Worlds interpretation, a world is a macroscopic state of all measuring devices, relative to a state of all measured systems; now we impose a final boundary condition by postselecting a macroscopic state. Why postselect a single macroscopic state? We could postselect a *superposition* of macroscopic states. For example, we could postselect a superposition of two macroscopic states, and get a "Two Worlds" interpretation of quantum theory. However, we have no reason to select such a superposition, for it would decohere. (See Sect. 9.4.) Recall the role that decoherence plays – what it can do and what it cannot do. Decoherence reduces an entangled state of measuring devices, measured systems, and their environment to a mixture of macroscopic states, i.e. to a mixture of states in which pointers all have well defined positions. Decoherence can thus determine the basis in which an entangled state reduces to a mixture. Decoherence *cannot* select a state (an actual measurement outcome) from the mixture (of possible measurement outcomes). But final boundary conditions can do that.

The paradoxes of Sect. 14.1, as well, motivate final boundary conditions. Figure 14.4 shows two particles in a singlet state $|\Psi_-\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. Alice, at spacetime point a, measures the spin component of one of them along some axis. In the Lorentz frame of Fig. 14.4, Alice's measurement fixes the spin component of the other particle at b. But in the Lorentz frame of Fig. 14.5, b precedes a; hence the spin component of the particle at b fixes the spin component of Alice's particle before she measures it. This paradox suggests that Alice's measurement at a is a future boundary condition on her particle before a.

So let *two* states describe the universe [7]. Formally, we have an initial or *history* state $|\Psi_{his}\rangle$ at time t_{in} and a final or *destiny* state $|\Psi_{des}\rangle$ at time t_{fin} as the initial and final boundary conditions on the universe. The times t_{in} and t_{fin} need not be the initial and final times of cosmology. Since the time evolution of $|\Psi_{his}\rangle$ and $|\Psi_{des}\rangle$ is unitary, t_{in} can be any time before the first measurement and t_{fin} can be any time after the last measurement. Let us choose, as an example of a history state at time t_{in}, a product of a single spin-1/2 state $a|\uparrow\rangle + b|\downarrow\rangle$, an initial state $|0\rangle_d$ of a measuring device, and an initial state $|0\rangle_{env}$ of their environment. At time

[2] The SL and CSL models make testable predictions that differ from the predictions of quantum mechanics. (See Sect. 9.2.) Experiments have not confirmed these predictions; but if future experiments do so, the conclusion of this paragraph will be invalid.

t_1 the measuring device measures the spin state; the history state becomes

$$|\Psi_{his}(t_1)\rangle = a|\uparrow\rangle \otimes |\uparrow\rangle_d \otimes |0\rangle_{env} + b|\downarrow\rangle \otimes |\downarrow\rangle_d \otimes |0\rangle_{env} \ ,$$

where the states $|\uparrow\rangle_d$ and $|\downarrow\rangle_d$ of the measuring device correspond to the outcome of the measurement. (To simplify the example, we suppose that the only nonzero terms in the Hamiltonian are the terms for the interaction of the measuring device with the spin and with the environment.) The measuring device, which is macroscopic, is not isolated from the environment; after a short decoherence time it becomes entangled with the environment, so that at some later time t_2, the history state is

$$|\Psi_{his}(t_2)\rangle = a|\uparrow\rangle \otimes |\uparrow\rangle_d \otimes |\uparrow\rangle_{env} + b|\downarrow\rangle \otimes |\downarrow\rangle_d \otimes |\downarrow\rangle_{env} \ ,$$

where now the states $|\uparrow\rangle_{env}$ and $|\downarrow\rangle_{env}$ of the environment show the dependence of the environment on the outcome of the measurement. The states $|\uparrow\rangle_{env}$ and $|\downarrow\rangle_{env}$ represent the degrees of freedom of the environment, which are beyond experimental control. All informatiom about the relative phase of $|\uparrow\rangle_d$ and $|\downarrow\rangle_d$ is lost via decoherence of these degrees of freedom. Thus decoherence determines the basis of macroscopic states $|\uparrow\rangle_d$ and $|\downarrow\rangle_d$ of the mixture that evolves from the measurement. For the destiny state at time $t_{fin} > t_2$ we define

$$|\Psi_{des}\rangle = |\uparrow\rangle_d \otimes |\uparrow\rangle_{env} \ .$$

The destiny state determines the states of the measuring device and the environment, i.e. the states that record the outcome of the measurement at time t_2. (For simplicity, we can assume that the measuring device is not used again after time t_2.) The destiny state selects from the history state at time t_2 just the state $|\uparrow\rangle_d \otimes |\uparrow\rangle_{env}$ of the measuring device and the environment. (The final state of the spin may depend on subsequent interactions with other measuring devices.) In this way, final boundary conditions on the universe complete the account of the measuring process.

The previous section discusses the axiom of stable collapse. If this axiom holds, quantum measurements will follow the Born rule. If it does not hold, measurements will not always follow the Born rule. (See Prob. 18.2.) So a failure of the Born rule would be evidence for unstable collapse and "improbable" final boundary conditions. Also, if weak measurements on a *pre*selected, but not *post*selected, ensemble yield weak values, we would interpret them as evidence for "improbable" final boundary conditions.

The two-state formalism is a minimal resolution of the paradox of Schrödinger's cat. It does not require a collapse mechanism. It simply equalizes the status of initial and final boundary conditions on the universe. We conclude from the discussion in Sect. 10.5, however, that the final boundary conditions are qualitatively different from the initial boundary conditions. The state $|\Psi_{des}(t_{fin})\rangle$ records, in one fashion or another, the result of every measurement ever made. The record could include pointers on measuring devices, notations in a laboratory notebook, and memories. The records could even be lost; but under unitary time evolution, no information is lost. Thus in $|\Psi_{des}(t_{fin})\rangle$, systems that once interacted and have since separated remain entangled. By contrast, initial boundary conditions do not record anything; the initial state $|\Psi_{his}(t_{in})\rangle$ is not entangled. The evolution is reversible, but not symmetric, in time.

Final boundary conditions resolve also the paradox of Lorentz invariance and collapse. (See Chap. 14.) In the two-state formalism, uncertainty about the outcome of a measurement is due to our ignorance of final boundary conditions. Hence our ignorance about the outcome of a measurement is like our ignorance about how a coin falls – "heads" or "tails" – before we look at it. There is no nonlocal collapse, because there is no collapse (only a final boundary condition); consequently, there is also no conflict with Lorentz invariance. This resolution of the paradoxes of Chaps. 9 and 14 extends automatically to relativistic quantum theory.

This resolution is consistent with causality only because most of $|\Psi_{des}\rangle$ is unknown, most of the time. In this respect, the two-state formalism is similar to Bohm's theory of hidden variables. Bohm's hidden variables *must* be hidden because otherwise we could use them to send superluminal signals. Similarly, if the destiny state were not partly hidden, we could send superluminal signals. For example, let Alice and Bob share a pair of spins in the state $(|\uparrow\rangle_A \otimes |\uparrow\rangle_B + |\downarrow\rangle_A \otimes |\downarrow\rangle_B)/\sqrt{2}$, where $|\uparrow\rangle_A$, $|\downarrow\rangle_A$ are states of a spin in Alice's laboratory and $|\uparrow\rangle_B$, $|\downarrow\rangle_B$ are states of a spin in Bob's. Alice and Bob adopt the following protocol: at time $t = 0$ (as they measure time) Alice may or may not flip her spin. Next, Bob checks whether his spin is in the state $|\uparrow\rangle_B$ or $|\downarrow\rangle_B$. Suppose that the destiny state shows the subsequent state of their spins to be $|\uparrow\rangle_A \otimes (|\uparrow\rangle_B + |\downarrow\rangle_B)/\sqrt{2}$. If Alice could anticipate this part of the destiny state, she could deliberately change the outcome of Bob's measurement by flipping her spin; thus she could send Bob a superluminal signal. From this example we see that causality requires the destiny state to be at least partly hidden.

The two-state formalism and Bohm's theory of hidden variables share another property: determinism. But the determinism of the two-state formalism is radically different from determinism as usually understood, for while $|\Psi_{his}\rangle$ evolves deterministically forward in time, $|\Psi_{des}\rangle$ evolves deterministically backward in time. Neither $|\Psi_{his}\rangle$ nor $|\Psi_{des}\rangle$ alone has any effect; an effect is the product of causal chains that extend from both the future and the past. In this formalism, old paradoxes of free will and determinism have a new solution. On the one hand, free will has to mean that our choices are not determined by our past. What is free will, if not freedom from the past? Such freedom has a place in the two-state formalism, because the history state $|\Psi_{his}\rangle$ by itself indeed does not determine the present. On the other hand, no one can expect to be free of the future. In the two-state formalism the past does not determine the future, yet the future is determined. This juxtaposition of free will and determinism recalls the classic Hebrew aphorism, "All is foreseen, yet choice is given" [8]; see also the work of Price [9].

18.4 The Role of $\hbar$

In searching for simple, physical axioms for quantum theory, we work from within quantum theory and from without. From within quantum theory, we seek a mechanics that directly expresses its physical content; and from without, we seek axioms that imply, directly or indirectly, just this physical content. In this section we work from within, formulating a mechanics in which modular variables play a role; in Sect. 18.6 we work from without, finding axioms that imply modular variables in this role.

Can there be a quantum theory without quantum states? Every quantum state is the eigenstate of some operators; we can therefore map a state to a set of operators. Now consider the

set of all Hermitian eigenoperators of a state $|\psi\rangle$, i.e. all Hermitian operators having $|\psi\rangle$ as an eigenstate. The set is algebraically closed in the sense that if A_ψ and A'_ψ are eigenoperators of the state $|\psi\rangle$, then so are $A_\psi + A'_\psi$ and $A_\psi A'_\psi$. The unique eigenstate of all the operators in this set is $|\psi\rangle$. (See Prob. 18.3.) In this sense, we can replace $|\psi\rangle$ with the set of all A_ψ. We can also define the set of all operators that transform $|\psi\rangle$ to a state orthogonal to $|\psi\rangle$. These two sets span the space of all operators.

The equation of motion for A_ψ is the Heisenberg equation of motion

$$\frac{dA_\psi}{dt} = \frac{i}{\hbar}[H, A_\psi] + \frac{\partial A_\psi}{\partial t} \ , \tag{18.9}$$

so the value of A_ψ at one time determines its value at all times. Once measured, the A_ψ are known at all times; the operators in the second set are unknown at all times, because they all transform $|\psi\rangle$ into some state orthogonal to $|\psi\rangle$ and so their expectation values in the state $|\psi\rangle$ all vanish. The set of completely known operators is maximal in that any operator that commutes with all the A_ψ is already in the set.

Suppose now that the only observables that we measure, on a system in the state $|\psi\rangle$, are the eigenoperators A_ψ, i.e. those for which the eigenvalues are (or will be) completely known. If we restrict ourselves to observables in this set, we never disturb the measured system and there is no uncertainty in our measurements. The evolution of the operators is completely deterministic. The boundary conditions on this evolution are just the eigenvalues of the A_ψ, so there is no longer any need for the state of the system. Thus, by restricting measurements to the maximal set of A_ψ, we eliminate states, uncertainties and probabilities from quantum theory. The restriction to measurements of the A_ψ is, of course, artificial. We might guess that there is no quantum theory left in this restricted theory; there is no collapse, and all that remains of the dynamics is the equation of motion, Eq. (18.9), which is the formal analogue of the classical equation of motion. Yet this restricted theory will surprise us: it is still quantum theory!

To see how the restricted theory works, consider a Hamiltonian H for a spinning particle constrained to the z-axis:

$$H = p_z^2/2m - g(t)\mu S_z B z \ .$$

For simplicity we take $g(t)$ to be impulsive: $g(t)$ vanishes except for a brief time interval $0 \le t \le T$ when it equals $1/T$. During this interval the magnetic field along the z-axis equals Bz, where B (like μ) is a constant.[3] The equations of motion obtained from H are

$$dz/dt = p_z/m \ , \quad dp_z/dt = g(t)\mu S_z B \ ,$$

and

$$dS_x/dt = g(t)\mu B z S_y \ , \quad dS_y/dt = -g(t)\mu B z S_x \ , \quad dS_z/dt = 0 \ .$$

If T is small, we can neglect the change in z during the interval $0 \le t \le T$ and calculate

$$p_z(T) = \mu S_z B + p_z(0) \tag{18.10}$$

[3] As in Sect. 7.2, the magnetic field must fringe away from the z-axis so as to satisfy Maxwell's equation $\nabla \cdot \mathbf{B} = 0$.

and

$$S_x(T) = S_x(0)\cos(\mu Bz) + S_y(0)\sin(\mu Bz) \ ,$$
$$S_y(T) = S_y(0)\cos(\mu Bz) - S_x(0)\sin(\mu Bz) \ ; \tag{18.11}$$

S_z does not change. Suppose that $p_z(0)$ is an approximate eigenoperator with eigenvalue 0, and $\Delta p_z(0)$ is small. Consider two possible initial boundary conditions for the spin. On one hand, if $S_z(T) = S_z(0)$ is one of the known observables, then from Eq. (18.10) we conclude that the particle accelerates in the direction of its spin, like atoms in a Stern-Gerlach apparatus. Equations (18.11) say nothing since $S_x(0)$ and $S_y(0)$ are completely unknown. On the other hand, if $S_x(0)$ is a known observable, we solve Eqs. (18.11) for $S_x(0)$ to obtain

$$S_x(0) = S_x(T)\cos(\mu Bz) - S_y(T)\sin(\mu Bz) \ . \tag{18.12}$$

Equation (18.12) indicates correlations between the modular position $z_{mod} = z \bmod 2\pi/\mu B$ and the spin components $S_x(T)$ and $S_y(T)$. For example, if at time T we find the particle at one of the points $z_{mod} = 0$, we are sure to find $S_x(T) = S_x(0)$. If at time T we find the particle at one of the points $z_{mod} = \pi/4\mu B$, we are sure to find $S_x(T) - S_y(T) = \sqrt{2}S_x(0)$. But so far the motion is classical, whether we choose $S_z(0)$ or $S_x(0)$ as the known observable.

The surprise comes when we consider modular angular momentum. The operator

$$e^{i\pi(L_x+S_x)/\hbar} \tag{18.13}$$

effects a rotation of π around the x-axis. This rotation, which sends $z \to -z$, $p_z \to -p_z$ and $S_z \to -S_z$, commutes with H and is thus a constant of the motion. It does not commute with S_z, so if $S_z(0)$ is one of the known observables, then the modular angular momentum $(L_x + S_x) \bmod 2\hbar$ is completely unknown. But it does commute with S_x, so if $S_x(0)$ is one of the known observables, then the modular angular momentum is known at all times.

Modular angular momentum is an observable with no classical analogue; it reveals nonlocal relative phases. For example, we know from Eq. (18.10) that $p_z(T)$ and S_z are correlated in sign, for $p_z(T) = p_z(T) - p_z(0) = \mu S_z B$. But how do we know, if we measure $S_x(0)$ and not S_z, about the relative phase between terms with different values of S_z and p_z? We can rephrase the question by returning for a moment to the Schrödinger formulation and specializing to spin-1/2. The initial state of the particle (up to normalization) is

$$\psi_+(z,0) = e^{-z^2/2(\Delta z)^2}\left(|\uparrow\rangle + |\downarrow\rangle\right) \ ,$$

where Δz is large. It evolves in time T to a superposition of two waves:

$$\psi_+(z,T) = e^{-z^2/2(\Delta z)^2 + i\mu Bz/\hbar}|\uparrow\rangle + e^{-z^2/2(\Delta z)^2 - i\mu Bz/\hbar}|\downarrow\rangle \ .$$

The superposed waves separate for $t > T$. How do we know, in our restricted theory, that the relative phase between the two terms in $\psi_+(z,T)$ is $+$ and not $-$ as in the orthogonal state

$$\psi_-(z,T) = e^{-z^2/2(\Delta z)^2 + i\mu Bz/\hbar}|\uparrow\rangle - e^{-z^2/2(\Delta z)^2 - i\mu Bz/\hbar}|\downarrow\rangle \quad ?$$

The answer is that the rotation operator of Eq. (18.13), which depends on the modular angular momentum $(L_x + S_x) \bmod 2\hbar$, reveals the relative phase. For all t, the modular angular

momentum of $\psi_+(z,t)$ equals $\hbar/2$, while the modular angular momentum of $\psi_-(z,T)$ equals $-\hbar/2$. Thus even this restricted theory, with its deterministic measurements, contains nonlocal quantum phases!

As long as $\hbar$ is nonzero, the theory contains nonlocal quantum phases. In the limit $\hbar \to 0$, modular angular momentum becomes undefined; the limit of Eq. (18.13) is undefined. No matter what boundary conditions we impose, modular angular momentum is completely unknown in this limit. The limit, then, gives us a new way to look at $\hbar$. The role of $\hbar$ is not quantitative (to measure uncertainty) but qualitative: $\hbar$ shows which observables become undefined in the classical limit.

18.5 Causality and Nonlocality as Axioms

Chapter 6 suggests two simple, physical axioms for quantum theory: relativistic causality and nonlocality. Does quantum theory follow from just these two axioms? This question gets us into a difficulty. The difficulty is that in *non*relativistic quantum mechanics, c cannot be a bound on the speed of signalling. We could try to derive *relativistic* quantum mechanics from these two axioms, but we get into another difficulty: relativistic quantum theory does not correspond satisfactorily to what experiments can measure. (See Sect. 11.5 and Chap. 14.) It makes more sense to try first to derive nonrelativistic quantum mechanics, as Chap. 6 suggests. But if we take the $c \to \infty$ limit of relativistic causality, we seem to be left with nothing – no limit on the speed of signalling.

Actually, the nonrelativistic theory itself shows the way out of the difficulty. Nonlocal quantum correlations are useless for signalling. (See Prob. 3.11.) Indeed, they *must* be useless for signalling. Consider two systems entangled in their internal degrees of freedom. The nonlocal correlations between the systems do not depend on their spatial separation; hence if the correlations were useful for signalling, they would be useful for superluminal signalling, too. So the causality constraint is *more* severe in the nonrelativistic limit: all signalling, not just superluminal signalling, is forbidden. Of course, signalling via local interactions (e.g. causal propagation of a field) is allowed, but signalling via any nonlocal interactions is forbidden. We can therefore reformulate the two axioms of Chap. 6 as follows:

i) Any nonlocal interactions are useless for signalling.

ii) Some interactions are nonlocal.

We must still define nonlocal interactions. What nonlocal interactions fit the second axiom best: Nonlocal correlations? Nonlocal equations of motion?

Suppose first that nonlocality means nonlocal correlations. Local correlations obey the CHSH inequality, Eq. (3.12). Namely, the sum of correlations

$$S_{CHSH}(A, A'; B, B') = C(A, B) + C(A', B) + C(A, B') - C(A', B')$$

satisfies $|S_{CHSH}(A, A'; B, B')| \leq 2$ if the correlations $C(A, B)$, etc. are local. The sum can be as large as $2\sqrt{2}$ if the correlations are quantum correlations [10]. Yet arbitrary correlations could yield a sum $|S_{CHSH}(A, A'; B, B')|$ as large as 4. Why does quantum mechanics limit the sum to $2\sqrt{2}$? Let's make a conjecture: the only way that $|S_{CHSH}(A, A'; B, B')|$ can

exceed $2\sqrt{2}$ is if the correlations violate causality. This conjecture, if true, would allow us to derive at least a part of quantum mechanics. We could deduce that, in order to satisfy the axiom of nonlocality (in the sense of nonlocal correlations), the sum $|S_{CHSH}(A, B; A', B')|$ would have to exceed 2 for some A, B, A' and B'; but in order to satisfy the axiom of causality, $|S_{CHSH}(A, A'; B, B')|$ could not exceed $2\sqrt{2}$. But a simple counterexample disproves the conjecture [11]. (See Prob. 18.5.) Deriving the $2\sqrt{2}$ limit from causality requires additional axioms [12].

Suppose now that nonlocality means nonlocal equations of motion. Are quantum equations of motion the only nonlocal equations of motion consistent with relativistic causality? Here, too, there is a counterexample, called *jamming*. In jamming, Alice and Bob jointly measure nonlocal correlations between two spacetime points a and b, respectively. But Jim (the jammer) can act at a spacetime point j to change their nonlocal correlations into local correlations. Jamming is nonlocal, but it is consistent with relativistic causality if a, b and j obey certain conditions [13]. (See Prob. 18.6.) So quantum mechanics is not the only theory combining causality with action at a distance.

Still, we continue the search for simple, physical axioms for quantum mechanics. So far we have discussed nonlocality in *space*. Bohm's theory of hidden variables, for example, is nonlocal across the spacelike separation between Bob's measurements and Alice's; the variables must be hidden to obey causality. The next section postulates a special form of action at a distance as the axiom of nonlocality. However, the final boundary conditions of Sect. 18.3 are hidden variables that are nonlocal in *time*. Section 18.3 shows that these nonlocal variables, as well, obey causality only if they are hidden. How does a theory combine causality with nonlocality in time?

Section 16.2 presents a thought experiment in which weak measurements of the kinetic energy of a particle consistently yield negative values. The thought experiment includes a final boundary condition – postselection of the particle far from the potential well in which it is initially bound. Instead of a postselection, we can follow the weak measurement of kinetic energy with a new and more precise measurement of kinetic energy (as Salviati points out in Sect. 16.8). If we do, the measured values are very likely to be positive. The fact that we may conclude the experiment with *either* a postselection *or* a new measurement of kinetic energy implies that there must be uncertainty in the measuring device and the measured system. That is, suppose the measuring device does not introduce uncertainty into the measured values. Then the scatter in (positive and negative) measured values must arise from uncertainty in the measured system. But even if the measuring device introduces uncertainty, there must also be uncertainty in the measured system. Otherwise, postselection on the measured system could not change the distribution of values measured by the device. Thus nonlocality in time implies fundamental uncertainty.

This derivation of uncertainty is qualitative, but a quantitative derivation may be possible as well. How much uncertainty is needed for nonlocality in time to be consistent with causality? The question remains open. We see again, however, that the positive in quantum mechanics hides behind the negative. Quantum uncertainty is not pointless "playing with dice", as Einstein put it; quantum uncertainty is the necessary and sufficient condition for time symmetry and causality to coexist.

18.6 Causality, Nonlocality and Scaling

Section 6.1 describes a piston in a long, closed cylinder with a particle in it. A little ball collides twice with the piston from the outside, pushing the piston inward. According to classical mechanics, only if the particle inside hits the piston does the ball affect it. But according to quantum mechanics, the ball affects the particle, even when there is no explicit interaction between them: it changes the modular energy of the particle. The particle does not, however, affect the ball. For our axiom of nonlocality, let us construct a model with the same nonlocal interaction. Let us assume that the ball changes the probability distribution of the energy of the particle, although there is no explicit interaction between the ball and the particle; we do not assume anything about *how* the distribution changes, just that it changes. The probability distribution of the energy of the ball, however, does not change.

According to our axiom of causality, the nonlocal interaction must be useless for signalling. But if the ball can affect the particle, why can't two observers, Alice and Bob, use it to send signals? If Bob, standing near the piston, scatters a ball off the piston, can't Alice, standing at the other end of the cylinder, observe a change in the probability distribution of the energy of the particle? Section 6.4 hints how to avoid a violation of causality: the axioms are compatible if and only if uncertainty keeps Alice from detecting the effect of the ball on the particle.

Let us apply a mathematical statement proved in Sect. 6.3. Let ρ_b and ρ_p denote the probability distributions for the energies of the ball and the particle, respectively, and ρ_{bp} the probability distribution for their total energy. The distribution of total energy is a convolution of ρ_b and ρ_p:

$$\rho_{bp}(E) = \int \rho_p(E - E')\rho_b(E')dE' \ . \tag{18.14}$$

We assume conservation of total energy, hence $\rho_{bp}(E)$ never changes. We also assume that the ball affects $\rho_p(E - E')$ without the particle affecting $\rho_b(E')$, so $\rho_p(E')$ is the only energy distribution that changes. The same must be true of the Fourier transforms of these distributions: $\tilde{\rho}_p(t)$ changes, but not $\tilde{\rho}_{bp}(t)$ or $\tilde{\rho}_b(t)$. From the Fourier transform of Eq. (18.14),

$$\tilde{\rho}_{bp}(t) = \tilde{\rho}_p(t)\tilde{\rho}_b(t) \ , \tag{18.15}$$

we see that for $\tilde{\rho}_p(t)$ alone to change, without any change in $\tilde{\rho}_b(t)$ and $\tilde{\rho}_{bp}(t)$, both $\tilde{\rho}_b(t)$ and $\tilde{\rho}_{bp}(t)$ must vanish over some interval in t. From Eq. (18.15) we then conclude, first, that the ball changes the *modular energy* of the particle (since the Fourier transform $\tilde{\rho}_p(t)$ represents the distribution of modular energy); and second, that the modular energy of the ball is completely unknown (since the Fourier transform $\tilde{\rho}_b(t)$ vanishes). If the modular energy of the ball is completely unknown, the change in modular energy of the particle does not correspond to any measurable change in the modular energy of the ball. Then the ball can affect the particle without the particle affecting the ball.

In quantum mechanics, the t in $\tilde{\rho}_p(t)$, $\tilde{\rho}_b(t)$ and $\tilde{\rho}_{bp}(t)$ necessarily represents time. (See Eq. 6.10.) Here t has (so far) nothing to do with time. Can we *deduce* that t represents time?

We need an additional axiom. Ideally, we would like an axiom that connects energy and time and holds both in classical and quantum mechanics. We note the following: Let H be a Hamiltonian for a system and O be any dynamical variable of the system. If we scale the

Hamiltonian by a factor λ, so that $H \to \lambda H$, then – whether the equation of motion for O follows from a Poisson bracket or a commutator – the corresponding scaling for dO/dt is $dO/dt \to \lambda dO/dt$. Let this be our third axiom: for any system, time derivatives of dynamical variables scale the same way as the Hamiltonian.

As an application of this axiom, let us compare two cases. In the first case, the probability distribution of the ball is $\rho_b(E)$. In the second case, the probability distribution is $\rho'_b(E)$, scaled by a factor λ relative to the first case:[4]

$$\rho'_b(E) = \frac{1}{\lambda}\rho_b\left(E/\lambda\right) \ . \tag{18.16}$$

If, in the first case, the time between the two collisions of the ball with the piston is T, what is the time between the two collisions in the second case? In the second case it is $T' = T/\lambda$, because the time evolution is faster by a factor λ.

Now let us compare the Fourier transforms of these probability distributions. The Fourier transform of $\rho_b(E)$ is

$$\tilde{\rho}_b(t) = (2\pi)^{-1/2} \int \rho_b(E) e^{iEt} dE \ .$$

The Fourier transform 0of $\rho'_b(E)$ is

$$\begin{aligned} \tilde{\rho}'_b(t) &= (2\pi)^{-1/2} \int \rho'_b(E) e^{iEt} dE \\ &= (2\pi)^{-1/2} \int \rho_b(E/\lambda) e^{i(E/\lambda)\lambda t/k} dE/\lambda \\ &= \tilde{\rho}_b(\lambda t) \ . \end{aligned} \tag{18.17}$$

What does Eq. (18.17) tell us? It tells us how to compare modular energy scales. The scale for $\tilde{\rho}'_b(t)$ is compressed by a factor λ compared to the scale for $\tilde{\rho}_b(t)$. In particular, suppose that $\tilde{\rho}_b(t)$ vanishes for all $t > t_0$. Equation (18.17) then implies that $\tilde{\rho}'_b(t)$ vanishes for all $t > t_0/\lambda$. In terms of modular energy, if in the case of $\rho_b(E)$ the modular energy E mod E_0 is completely uncertain for all $E_0 < 2\pi/t_0$, then in the case of $\rho'_b(E)$ the modular energy E mod E_0 is completely uncertain for all $E_0 < 2\pi\lambda/t_0 = 2\pi(T/t_0)/T'$ (since $\lambda = T/T'$). If we fix T and define a constant $k = T/t_0$, then the modular energy E mod E_0 is completely uncertain for all $E_0 < 2\pi k/T'$.

We thus obtain an uncertainty relation for energy and time. For if E mod E_0 is completely uncertain, then the uncertainty ΔE in energy cannot be less than E_0. Thus the minimum uncertainty in the energy is related to the time T' between the two collisions of the ball by $\Delta E \geq 2\pi k/T'$, or

$$T'\Delta E \geq 2\pi k \ . \tag{18.18}$$

Equation (18.18) has the form of the uncertainty relation for energy and time in quantum theory. (See Sects. 8.1 and 8.5.) But is the constant k universal? This question takes us back to Sects. 2.3–4 and the Bohr-Einstein debate. It might seem that Eq. (18.18) holds only when

[4]The normalization is the same: $\int \rho'_b(E)dE = \int \rho_b(E)dE$.

the energy distribution of the ball has the special form $\rho'_b(E)$, defined in Eq. (18.16), for some value of λ. But Eq. (18.18) is a consequence of the axiom of causality: it prevents signalling between systems that do not interact locally. It must therefore hold for all possible energy distributions of the ball; otherwise we could defeat the axiom of causality, as it applies to one distribution, with another distribution. The Bohr-Einstein debate shows that uncertainty relations hold consistently only if they hold universally. The theory that follows from our three axioms can be consistent only if the constant k is the same for all energy distributions. Since the ball could arrive at the piston after colliding with any other system, the constant k in the uncertainty relation Eq. (18.18) must be universal, and we can identify k with $\hbar$. Our three axioms thus imply at least a part of quantum theory.

18.7 What is the Quantum World?

The title of this chapter is "The Quantum World". If Bohr could have reviewed the book, he might [14] (or might not [15]) have objected, "There is no quantum world. There is only an abstract quantum physical description." There is no quantum world independent of our measurements.

But – to paraphrase Maslow [16] – if our only tool is a hammer, we tend to treat everything as if it were a nail. If our only tool were a hammer, not even the classical world would be independent of our measurements. This book proposes measurements that do not hammer at the quantum world, and claims that the quantum world is independent of them. Let us briefly review these measurements.

If a system is in an eigenstate $|\psi\rangle$ of an observable A, a measurement of A on the system yields an eigenvalue of A and the measurement does not disturb the measured system. Section 18.4 makes the most of such measurements by defining the set of all observables A_ψ having $|\psi\rangle$ as an eigenstate, and considering measurements of the A_ψ only. If we measure only eigenoperators of $|\psi\rangle$, there is no uncertainty in our measurements and we never disturb the measured system. We might expect that nothing quantum could come out of these measurements. Yet among the operators A_ψ are modular variables; as Chaps. 5 and 6 show, modular variables represent nonlocal quantum phases. In the limit $\hbar \to 0$ modular variables are completely undefined. Hence the classical limit $\hbar \to 0$ is *not* a limit in which all observables are well defined, although conjugate variables are well defined. For $\hbar \neq 0$, conjugate variables do not commute but modular variables are well defined; and measurements of modular variables exhibit quantum nonlocality, part of the quantum world that other measurements miss.

If $|\psi\rangle$ is not an eigenstate of an operator A, we can still measure A in a protective measurement. In such a measurement, the coupling between A and the measured system is weak and $|\psi\rangle$ is protected from any disturbance due to this weak coupling. The protection may, for example, consist of many measurements of $|\psi\rangle\langle\psi|$ during the measurement of A. As Sect. 15.3 shows, the measurement yields $\langle\psi|A|\psi\rangle$, the average of A measured on an ensemble of systems in the state $|\psi\rangle$. Indeed, a protective measurement on a *single* system yields $\langle\psi|A|\psi\rangle$. The operator A is arbitrary; hence we can measure every relevant observable on a single system. Quantum theory is not merely a set of rules for calculating probabilities. Protective measurements show that the magnitude and phase of quantum waves, and expectation values such as $\langle\psi|A|\psi\rangle$ and

$|\langle a|b\rangle|^2$ (i.e. the expectation value of $|b\rangle\langle b|$ in the state $|a\rangle$) have physical meaning independent of probabilities.

Weak measurements generalize protective measurements. In weak measurements, the coupling is weak; but a weak measurement of A involves preselection of a state $|\psi_{in}\rangle$ and postselection of a state $|\psi_{fin}\rangle$. These states define a pre- and postselected (PPS) ensemble, and the measured value, the weak value $\langle A\rangle_w$, has the form $\langle\psi_{fin}|A|\psi_{in}\rangle/\langle\psi_{fin}|\psi_{in}\rangle$. Weak values lead us to reinterpret the role of time in quantum mechanics. For if a weak value describes a quantum system between two times, then the future of a quantum system must contain information about its present that is not available in its past. We come to describe quantum systems via two states, one evolving forwards from the past and the other evolving backwards from the future, as in Sect. 18.3.

The time symmetry in this new formalism is elegant, but it is not only elegant: time symmetry is crucial for the definition of the quantum world. It is time symmetry that allows us to couple a measuring device to an independent quantum system, as the "quantum walk" of Sect. 16.6 shows. During a weak measurement, the measured system and measuring device evolve together to a highly entangled state, Eq. (16.21). Without postselection there is no way to disentangle the entanglement of the measured system and measuring device, hence no quantum world. *With* postselection, however, the measuring device measures a single value – the weak value – that depends only on the measured system and not on the measuring device. (See Eqs. (16.22–24).) *With* postselection, there is no superposition of pointer positions; there is no need to monitor quantum phases in the measuring device. The measuring device decouples from the measured system, and we can, with Bohr, treat it as classical.

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Such is the world to which quantum paradoxes have led us. And after all, we look forward to an application: may this book inspire readers to confront paradox creatively, "to think for themselves, to unmask false arguments and ambiguous phrases" [17] in the world, and to grasp the world more fully by grasping it gently.

Problems

*18.1 (a) Define a state $|\psi\rangle$ of a two-level system

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle$$

where $\langle i|j\rangle = \delta_{ij}$. Let A be the operator $|1\rangle\langle 1| - |2\rangle\langle 2|$. Compute the expectation value $\langle A\rangle$ and the uncertainty $\Delta A = [\langle A^2\rangle - \langle A\rangle^2]^{1/2}$ of A in the state $|\psi\rangle$ and show that

$$A|\psi\rangle = \langle A\rangle|\psi\rangle + \Delta A|\psi_\perp\rangle \ ,$$

where $\langle\psi|\psi_\perp\rangle = 0$. Write down $|\psi_\perp\rangle$ explicitly.
(b) Let $|\Psi_{in}\rangle$ be an initial product state of N identical systems prepared in the state $|\psi\rangle$:

$$|\Psi_{in}\rangle = \bigotimes_{i=1}^{N} |\psi\rangle_i \ .$$

Consider a weak measurement of $\bar{A} = \sum_i A^{(i)}/N$ on this initial state (where $A^{(i)}$ represents the operator A on the i-th system) assuming that the time evolution operator for the systems, over the duration of the measurement, is $e^{-iP_d\bar{A}/\hbar}$. (As usual, P_d is conjugate to the position Q_d of the pointer on a measuring device.) Show that the state of the systems immediately after the weak measurement, up to terms of order $1/N^2$, is

$$\left[\cos\frac{P_d}{\hbar N} - i\langle A\rangle \sin\frac{P_d}{\hbar N}\right]^N |\Psi_{in}\rangle - i(\Delta A)\sin\frac{P_d}{\hbar N}\sum_{i=1}^{N}|\Psi_i^{\perp}\rangle\ , \tag{18.19}$$

where $|\Psi_i^{\perp}\rangle$ represents the i-th system in the state $|\psi_\perp\rangle$ and all the other systems in the state $|\psi\rangle$.
(c) Now assume that, immediately following the weak measurement, an exact measurement of A on each system yields N_1 of the systems in the state $|1\rangle$ and N_2 in the state $|2\rangle$. Show that if $N_1 = N$ or $N_2 = N$, then all the terms in the sum in Eq. (18.19) add constructively and the time evolution operator moves the pointer by 1 or -1, respectively. Show that the terms in the sum cancel out in the limit $N \to \infty$ if and only if $N_1/N = c_1^* c_1$ in the same limit.

18.2 Consider an ensemble of N identical systems in the pre- and postselected states

$$|\Psi_{in}\rangle = |\uparrow\rangle\ , \quad |\Psi_{fin}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)\ .$$

The Hamiltonian of these systems does not depend on spin. On each system in this PPS ensemble we measure the operator

$$\mathbf{n}\cdot\sigma = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}\ ,$$

each time choosing the unit vector $\mathbf{n}$ at random. In the limit $N \to \infty$, what probability distribution describes these measurements? Show that the probability of obtaining spin parallel to $\mathbf{n}$ is not $\cos^2(\theta/2)$ (as it would be without postselection) but

$$\cos^2(\theta/2)\frac{1+\cos\phi\sin\theta}{1+\cos\phi\sin\theta\cos\theta}\ .$$

18.3 Let $|\psi\rangle$ be a state in an n-dimensional complex vector space with $n > 2$. Show that every Hermitian eigenoperator A_ψ of $|\psi\rangle$ is a real linear combination of $(n-1)^2+1$ linearly independent operators, and find an explicit representation for the $(n-1)^2+1$ operators. Show that the set of all these eigenoperators of $|\psi\rangle$ uniquely defines $|\psi\rangle$. Why is $n = 2$ a special case?

18.4 Consider two lines in the xy-plane, parallel to the x-axis. One line (the "high road") has $y = L$, while the other (the "low road") has $y = -L$. Two particles, A and B, move along these lines, in the direction of increasing x. On the y-axis, at the points $(0, L)$ and $(0, -L)$, sit two two-state "atoms". (See Fig. 18.2.) Let $|\uparrow\rangle_+$ and $|\downarrow\rangle_+$ denote

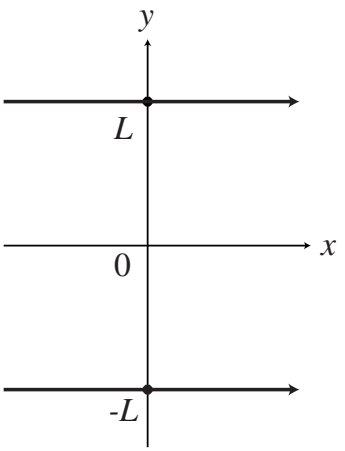

Figure 18.2: The "high road" passes through a two-state "atom" at $(0, L)$, and the "low road" passes through a two-state "atom" at $(0, -L)$, in the thought experiment of Prob. 18.4.

states of the atom at $(0, L)$ and $|\uparrow\rangle_-$ and $|\downarrow\rangle_-$ denote states of the atom at $(0, -L)$. (The atoms really have zero angular momentum, but this notation is convenient.) If particle A encounters one of these two-state atoms, it acts upon it as σ_z (multiplies $|\downarrow\rangle_+$ or $|\downarrow\rangle_-$ by -1); if particle B encounters one of the atoms, it acts upon it as σ_x (interchanges $|\uparrow\rangle_+$ and $|\downarrow\rangle_+$ or $|\uparrow\rangle_-$ and $|\downarrow\rangle_-$).
(a) Suppose the initial states of particles A and B are $[|+\rangle_A + |-\rangle_A]/\sqrt{2}$ and $[|+\rangle_B + |-\rangle_B]/\sqrt{2}$, respectively, where $|+\rangle$ denotes the high road and $|-\rangle$ denotes the low road; the initial state of the atoms is $[|\uparrow\rangle_+ \otimes |\uparrow\rangle_- + |\downarrow\rangle_+ \otimes |\downarrow\rangle_-]/\sqrt{2}$. Show that the final state of the atoms is $[|\uparrow\rangle_+ \otimes |\downarrow\rangle_- - |\downarrow\rangle_+ \otimes |\uparrow\rangle_-]/\sqrt{2}$, but the final state of particles A and B depends on which particle first encounters the atoms.
(b) Show that the atoms and the trailing particle exchange $L_x \bmod 2\hbar$ (modular angular momentum) nonlocally. (The operator $e^{iL_x\pi/\hbar}$ interchanges $|+\rangle_A$ and $|-\rangle_A$, $|+\rangle_B$ and $|-\rangle_B$, $|\uparrow\rangle_+$ and $|\uparrow\rangle_-$, and $|\downarrow\rangle_+$ and $|\downarrow\rangle_-$.)
(c) Assume that the initial states of particles A and B are $|+\rangle_A$ and $|-\rangle_B$, respectively, so the order in which the particles arrive at the atoms is not Lorentz-invariant. How can the atoms and the trailing particle exchange modular angular momentum nonlocally, when the trailing particle is different in different Lorentz frames? Resolve this paradox.

18.5 In an experiment to test Bell's inequality, Alice and Bob share pairs of spin-1/2 particles; Alice measures either $\mathbf{a} \cdot \sigma$ or $\mathbf{a}' \cdot \sigma$ on her particles while Bob measures either $\mathbf{b} \cdot \sigma$ or $\mathbf{b}' \cdot \sigma$ on his, where $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$ and $\mathbf{b}'$ are unit vectors in space. (See Sect. 3.4.) For pairs in a singlet state, the quantum correlation between a measurement of $\mathbf{a} \cdot \sigma$ by Alice and $\mathbf{b} \cdot \sigma$ by Bob is $C_Q(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$. Consider a "superquantum" correlation $C_{SQ}(\mathbf{a}, \mathbf{b})$ for pairs of spin-1/2 particles such that

$$C_{SQ}(\mathbf{a}, \mathbf{b}) = -\mathrm{sgn}(\mathbf{a} \cdot \mathbf{b}) \ ;$$

the measured correlations (and probabilities) depend only on the sign of $\mathbf{a} \cdot \mathbf{b}$.
(a) Show that $C_{SQ}(\mathbf{a}, \mathbf{b})$ is consistent with relativistic causality and that a suitable

choice of $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$ and $\mathbf{b}'$ yields

$$|C_{SQ}(\mathbf{a},\mathbf{b}) + C_{SQ}(\mathbf{a}',\mathbf{b}) + C_{SQ}(\mathbf{a},\mathbf{b}') - C_{SQ}(\mathbf{a}',\mathbf{b}')| = 4\ . \tag{18.20}$$

(b) Find a superquantum correlation $C_{SQ}(\mathbf{a},\mathbf{b})$ that is continuous and differentiable for all $\mathbf{a}$ and $\mathbf{b}$, consistent with relativistic causality, and yields Eq. (18.20) for a suitable choice of $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$ and $\mathbf{b}'$.

18.6 Assume that in the absence of jamming, correlations measured jointly at spacetime points a and b are the correlations of a singlet state of two spin-1/2 particles

$$(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B)/\sqrt{2}\ ;$$

but the effect of jamming at the spacetime point j is to turn these correlations into the correlations of an equal mixture of the states $|\uparrow\rangle_A \otimes |\uparrow\rangle_B$ and $|\downarrow\rangle_A \otimes |\downarrow\rangle_B$. Show that jamming is consistent with relativistic causality if and only if the intersection of the forward light cones of a and b lies entirely within the forward light cone of j.

18.7 Find a generalized density matrix $\rho(t)$ with the following property: tr $[\rho(t)A]$ is the weak value of A at time t, for any observable A of a system preselected in the state $|\Psi_{in}\rangle$ at time t_{in} and postselected in the state $|\Psi_{fin}\rangle$ at time t_{fin}. (See Prob. 9.6.) Show that $\rho(t)$ evolves in time as an ordinary density matrix:

$$\rho(t') = U(t'-t)\rho(t)U^\dagger(t'-t)\ ,$$

where $U(t'-t)$ evolves the system over a time $t'-t$. (Assume that the Hamiltonian of the system does not depend on time.)

*18.8 Consider a measurement sequence in four steps. The first step is preselection of the state

$$|\Psi_{in}\rangle = \bigotimes_{n=1}^{N} \frac{|\uparrow\rangle_n + |\downarrow\rangle_n}{\sqrt{2}}\ .$$

The second step is a measurement interaction of the form

$$\hbar P \sum_{n=1}^{N} (-1)^n \frac{1-\sigma_z^{(n)}}{2}\ ,$$

i.e. the unitary operator

$$U = e^{-iP\sum_n (-1)^n [1-\sigma_z^{(n)}]/2}$$

acting on $|\Psi_{in}\rangle$. The third step is a measurement of $\sum_n \sigma_x^{(n)}/N$, corresponding to the unitary operator

$$U_d = e^{-(i/\hbar) g_0 P_d \sum_n \sigma_x^{(n)}/N}\ .$$

For large N this measurement is naturally weak. The fourth step is postselection of the state

$$|\Psi_{fin}\rangle = \bigotimes_{n=1}^{N} |\uparrow\rangle_n \ .$$

Assume P and P_d are conjugate to pointer positions Q and Q_d, respectively. The pre- and postselected states could represent N spins; they could also represent N particles passing through a screen with two slits, as in Fig. 16.1. If they represent particles passing through a screen, these measurements recall the two-slit experiment of Sect. 18.1. In either case, the first, third and fourth steps alone define a weak measurement on a PPS ensemble, with the weak value of $\sum_n \sigma_z^{(n)}/N$ exhibiting the interference between $|\uparrow\rangle_n$ and $|\downarrow\rangle_n$ in $|\Psi_{in}\rangle$ even though there is no interference in the postselected state $|\Psi_{fin}\rangle$. Is it a paradox that we can exhibit interference without interference in the final state? We know that weak values depend on both the preselected state and the postselected state. (See Sect. 16.6.) Yet this measurement sequence, including all four steps, leads to a paradox.

(a) Show that

$$\langle\Psi_{fin}|U_d U|\Psi_{in}\rangle = e^{-(i/\hbar)g_0 P_d \cos P}$$

exactly in the limit $N \to \infty$.

(b) Let $\Phi_d(Q_d)\Phi(Q)$ be the initial state of the pointers, with $\Phi_d(Q_d) = (\epsilon^2\pi)^{-1/4} e^{-Q_d^2/2\epsilon^2}$. Suppose first that $\Phi(Q)$ is an eigenstate of P. That is, U is not a measurement but an operation that changes the relative phase between $|\uparrow\rangle_n$ and $|\downarrow\rangle_n$ in the preselected state $|\Psi_{in}\rangle$. The pointer Q_d measures this change in relative phase; it moves by $g_0 \cos P$. For example, if $P = \pi$, then U changes the relative sign between $|\uparrow\rangle_n$ and $|\downarrow\rangle_n$ from $+$ to $-$, and Q_d moves by $-g_0$ instead of g_0. The effect of U is paradoxical, because $\langle\Psi_{fin}|U = \langle\Psi_{fin}|$. It is true U_d comes between $\langle\Psi_{fin}|$ and U, but U_d represents a weak measurement; it should not affect the final state. Show, indeed, that $\langle\Psi_{fin}|U_d|\Psi_{fin}\rangle = 1$ in the limit $N \to \infty$; there are no spin flips $|\uparrow\rangle_n \leftrightarrow |\downarrow\rangle_n$ in this limit.

(c) But for this PPS ensemble, the relevant calculation is not $\langle\Psi_{fin}|U_d|\Psi_{fin}\rangle$ but $\langle\Psi_{fin}|U_d|\Psi_{in}\rangle$. Show that if $|\Psi_{in}\rangle$ were a typical state in which about half the spins pointed up, and about half down, the x-axis, then $2^{N/2}\langle\Psi_{fin}|U_d|\Psi_{in}\rangle$ would equal 1 in the limit $N \to \infty$; but $|\Psi_{in}\rangle$ is not a typical state and $2^{N/2}\langle\Psi_{fin}|U_d|\Psi_{in}\rangle$ equals $e^{-ig_0 P_d/\hbar}$.

(d) Calculate $\langle\Psi_{fin}|U_d|\Psi_{in}\rangle$ as a power series in g_0 by expanding the exponential in U_d. In this power series, terms of k-th order typically represent k spin flips between $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$. For each k, estimate the contribution of all terms of k-th order. (Approximate $k! \approx k^k$, etc.) Show that in the limit $N \to \infty$, the largest contribution comes from terms of order $k \approx g_0$. Thus in the limit $N \to \infty$, the *fraction* of spins flipped vanishes but the *number* does not. If $|\uparrow\rangle_n$ and $|\downarrow\rangle_n$ represent states of a particle passing through a screen with two slits, the weak measurement changes the final direction of approximately $g_0 P_d/\hbar$ of the particles.

(e) Suppose now that U is a measurement; let $\Phi(Q)$ be a gaussian state in which the expectation value of P vanishes, and ΔP is small. Then U does not change the relative phases in $|\Psi_{in}\rangle$, and during the U_d measurement, Q_d moves by g_0 (to order ΔP). Show that Q does not move at all, but the uncertainty in Q increases from ΔQ in the initial state to $[(\Delta Q)^2 + \hbar^2(g_0\Delta P_d)^2/4(\Delta Q)^2]^{1/2} \geq \sqrt{\hbar g_0 \Delta P_d}$. Show that the uncertainty in the measured value of P due to the uncertainty in Q_d is approximately $\sqrt{\Delta Q_d/g_0}$. Thus the U_d measurement decreases the uncertainty in P, but the uncertainty in Q increases such that the product of the uncertainties is not less than $\hbar/2$.

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